Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 14, No. 1, 2026, pp. 50-59
DOI:10.22034/cmde.2025.64394.2921

Investigation of convergence analysis of the stochastic Heston model with one singular point

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Abstract

The Heston model is a popular stochastic volatility model used in financial mathematics for option pricing. This paper focuses on the stochastic Heston model (SHM) with one singular point. In this way, we first consider the existence, uniqueness, and boundedness of the numerical solution under the global Lipschitz condition and the linear growth condition. In addition, the stochastic θ-scheme is developed to solve the equation numerically, and we obtain a convergence rate with $\min\{2-2\alpha, 1-2\beta\}$ which depends on the kernel parameters. Moreover, Monte Carlo (M.C.) simulation is implemented for this kind of problem in the 95 percent confidence interval, which reveals that it verifies the stochastic θ-scheme results. Finally, a numerical example is given to show the validity and effectiveness of the theoretical results.

Keywords. Stochastic Heston model, Singular point, Convergence analysis, Existence and uniqueness, Option pricing, Monte Carlo simulation.
2010 Mathematics Subject Classification. 45D05, 45G05, 60H20, 65C30.

1. Introduction

The rough Heston model belongs to the broader class of rough volatility models. These models are characterized by incorporating rough paths or fractional Brownian motion (FBM) to capture the long-term memory and persistence of volatility in financial markets. The rough Heston model, with its unique set of parameters and characteristics, offers a distinct framework for modeling and understanding volatility dynamics in financial markets [14, 24, 25]. In this paper, we consider the following stochastic Heston model (SHM) with one singular point:

$$\begin{cases} dS(t) = S(t)\sqrt{V(t)}dW(t), \\ V(t) = V_0 + \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^t (t-s)^{-\alpha}\big(\omega(s) - \lambda V(s)\big)ds + \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^t (t-s)^{-\beta}\nu\sqrt{V(s)}dB(s), \quad t\in[0,T], \end{cases} \tag{1.1}$$

where

- $W(t) := \big(W_1(t), W_2(t), \dots, W_r(t)\big)^T, 0 \le t \le T$ be an r-dimensional standard Brownian motion defined on the complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a given $T > 0$.
- $V_0 \in \mathbb{R}^d$ is an $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\big(|V_0|^2\big) < +\infty$ and $V(s) := \lim_{r\uparrow s} V(r)$.

The parameters λ, ω, V_0, and ν are all positive and γ belongs to $(\frac{1}{2}, 1)$. The stochastic process $W(t)$ is defined as:

$$W(t) = \rho B(t) + \sqrt{1-\rho^2}B^{\perp}(t), \tag{1.2}$$

where $\rho \in [-1, 1]$ indicates the correlation coefficient and $\big(B(t), B^{\perp}(t)\big)$ represents a two-dimensional fractional Brownian motion.

Stochastic Volterra integral equations (SVIEs) were first proposed in [8, 9], and can be regarded as generalizations of stochastic differential equations (SDEs) [5–7], or deterministic Volterra integral equations. These kinds of equations are very popular in several areas of science. For example, they are frequently used for modeling biological mathematics

Received:07 November 2024; Accepted: 19 February 2025.
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[19, 23], financial market [28, 34], control science [1], and medicine [27]. Meanwhile, many researchers focused on the numerical analyses for SVIEs [3, 4, 18]. An area of particular interest in the study of SVIEs is stochastic numerical analysis, since the closed-form solution of SVIEs is rarely known and stochastic numerical approximations provide a robust algorithm for understanding solution behavior. So far, most numerical approximations have been developed to deal with SVIEs with regular kernels. For example, Xiao et al. [31] introduced a split-step collocation method for SVIEs. Liang et al. [22] presented the Euler–Maruyama method for linear stochastic Volterra integral equations of strong convergence with order $\frac{1}{2}$. Conte et al. [10] introduced the improved stochastic θ-methods for the numerical integration of stochastic Volterra integral equations.

SIVE with singular kernels, however, appear in the treatment of some problems in the field of stochastic partial differential equations and in the analysis of fractional Brownian motion [30]. Zhang [33] investigated the convergence of the EM scheme and large deviations for stochastic Volterra equations with singular kernels. Xiao [32] obtained the convergence rate of the EM method for SVIEs with Abel-type kernels under global Lipschitz condition and linear growth condition. Wang [29] established the existence and uniqueness theorem under non-Lipschitz conditions, plus linear growth conditions as well as certain integrable conditions.

Very often, the singular SVIEs do not admit closed-form solutions, and thus, they require some discretization techniques. Since the singularity of the integrand kernel leads to greater difficulties both at the point $s = 0$ and at the point $s = t$. In this case, the important Itô formula, which is a powerful and necessary tool in the study of SDEs, is not available, and we have to look for other tools and techniques. For example, Li et al. [20] investigated the exact asymptotic separation rate of two distinct solutions of doubly singular stochastic Volterra integral equations with two different initial values using the Gronwall inequality. Li et al. [21], concerned with the more general nonlinear stochastic Volterra integral equations with doubly singular kernels, proposed a Galerkin approximation scheme to solve the equation numerically, and obtained the strong convergence rate for the Galerkin method in the mean square sense. Dai and Xiao [11] consider Levy noise-driven nonlinear stochastic Volterra integral equations with doubly weakly singular kernels and uses a fast EM method presented to overcome the low computational efficiency of the EM method.

To solve the singular SHM (1.1) numerically, one can develop some high-order numerical methods. Unfortunately, the important Itô formula for Eq. (1.1) is not available. Therefore, this paper will focus mainly on the following two objectives:

- Under the global Lipschitz and the linear growth conditions, the first target of this paper is to examine the existence, uniqueness, and boundedness of the numerical solution.
- Propose the stochastic θ-scheme for SHM with one singular point, and figure out the strong convergence order with $\min\{2-2\alpha, 1-2\beta\}$ under the global Lipschitz condition and the linear growth condition, and we can enhance the speed of convergence by changing the parameters θ.

The remainder of the paper is organized as follows. Section 2 presents some necessary notations and assumptions. Section 3 aims to derive the existence, uniqueness, and boundedness of the numerical solution and strong convergence properties of the proposed scheme. Section 4 presents numerical experiments showing perfect accordance with the theoretical results established in section 3. In section 5, there is a brief statement as the conclusion of this paper.

2. Mathematical Preliminaries

Throughout this paper, we will use $|\cdot|$ as the Euclidean norm on $\mathbb{R}^d$, that is, for $V \in \mathbb{R}^d$, then $|V| = \left(\sum_{i=0}^{d} V_i^2\right)^{\frac{1}{2}}$; and $\|\cdot\|$ denotes the trace norm of a matrix, that is to say that for $A \in \mathbb{R}^{d\times r}$, $\|A\| = \sqrt{trace(A^T A)}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual assumptions (i.e. it is right continuous, increasing and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets). In addition, the capital letter C is used to represent a generic positive constant whose value can change when it appears in several places, but it is always independent of the step size h.

Assumption 2.1. *(Global Lipschitz condition): There exists a positive constant L, such that for all $V_1, V_2 \in \mathbb{R}^d$, the following inequality holds:*

$$|f(V_1) - f(V_2)|^2 \vee \|g(V_1) - g(V_2)\|^2 \leq L|V_1 - V_2|^2. \tag{2.1}$$

Assumption 2.2. *(Linear growth condition): There exists a positive constant K, such that for all $V \in \mathbb{R}^d$, the following inequality holds:*

$$|f(V)|^2 \vee \|g(V)\|^2 \leq K(1+|V|^2). \tag{2.2}$$

Definition 2.3. ([13]): Assume that $\alpha \in (0,1]$, $T \in [0,+\infty)$ and $f:[0,T] \to \mathbb{R}^d$ be a measurable function such that $\int_0^T |f(\tau)|d\tau < +\infty$. The Riemann–Liouville fractional integral operator of order α is defined as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \forall\, t \in [0,T], \tag{2.3}$$

where $\Gamma(\alpha)$ is the Gamma function.

Remark 2.4. If the positive constant α satisfies $\alpha \in (0,1)$, then for all $t \in [t_n, t_{n+1}]$, $n = 1,2,\cdots,N-1$ we have:

$$\int_0^{t_n} (t_n - s)^{-\alpha} ds = \frac{1}{1-\alpha} t_n^{1-\alpha}. \tag{2.4}$$

3. Convergence rate of stochastic θ-scheme

Since it is difficult to obtain the closed-form solution of the SHM with one singular point (1.1), considering effective numerical methods becomes particularly necessary.

3.1. Stochastic θ-approximation.

For every integer $N \geq 1$, the stochastic θ-approximation can be given as follows:

$$\begin{aligned} V^N(t) &= V_0 + \theta \int_0^t (t-s)^{-\alpha} f(\check{V}^N(s))ds + (1-\theta)\int_0^t (t-s)^{-\alpha} f(\hat{V}^N(s))ds \\ &\quad + \int_0^t (t-s)^{-\beta} g(\hat{V}^N(s))dW(s), \qquad t \in [0,T], \end{aligned} \tag{3.1}$$

where $\check{V}^N(s) := V^N(t_{n+1})$ for $s \in (t_n, t_{n+1}]$ and $\hat{V}^N(s) := V^N(t_n)$ for $s \in [t_n, t_{n+1})$. Here, $t_n = nh$ $(n = 0,1,\ldots,N)$ denote the grid points with step size $h = T/N$.

3.2. Existence and uniqueness theorem.

In this section, we delve into the existence and uniqueness of the proposed method.

Lemma 3.1. *Let the positive constants α and β satisfy $\alpha \in (0,1)$ and $\beta \in (0,\frac{1}{2})$. If the nonlinear functions f and g satisfy the linear growth condition, then, there exists a positive constant C independent of N for given $T>0$, such that:*

$$\sup_{0\leq t\leq T} \mathbb{E}\big(|\check{V}^N(t)|^2\big) \vee \sup_{0\leq t\leq T} \mathbb{E}\big(|\hat{V}^N(t)|^2\big) \leq \sup_{0\leq t\leq T} \mathbb{E}\big(|V^N(t)|^2\big) \leq C. \tag{3.2}$$

Proof. For every integer $k \geq 1$, we define the stopping time as follows:

$$\tau_k = T \wedge \inf\{t \in [0,T] : |V^N(t)| \geq k\}, \tag{3.3}$$

where $\tau_k \uparrow T$ almost surely as $k \to +\infty$. For simplicity, set $V_k^N(t) = V^N(t\wedge\tau_k), \check{V}^N(t) = \check{V}^N(t\wedge\tau_k)$, and $\hat{V}^N(t) = \hat{V}^N(t\wedge\tau_k)$ for $t \in [0,T]$. Then, $V_k^N(t)$ satisfies the equation:

$$\begin{aligned} V_k^N(t) &= V_0 + \theta \int_0^{t\wedge\tau_k} (t\wedge\tau_k - s)^{-\alpha} f(\check{V}_k^N(s))ds + (1-\theta)\int_0^{t\wedge\tau_k} (t\wedge\tau_k - s)^{-\alpha} f(\hat{V}_k^N(s))ds \\ &\quad + \int_0^{t\wedge\tau_k} (t\wedge\tau_k - s)^{-\beta} g(\hat{V}_k^N(s))dW(s). \end{aligned} \tag{3.4}$$

Now by using the power mean inequality, Itô isometry, Cauchy-Schwarz's inequality and the linear growth condition (2.2), we have:

$$\mathbb{E}\big(|V_k^N(t)|^2\big) \leq 4\mathbb{E}\big(|V_0|^2\big) + 4\theta^2 \mathbb{E}\Big(\int_0^{t\wedge\tau_k} (t\wedge\tau_k - s)^{-\alpha} ds \int_0^{t\wedge\tau_k} (t\wedge\tau_k - s)^{-\alpha} |f(\check{V}_k^N(s))|^2 ds\Big)$$

$$+ 4(1-\theta)^2\mathbb{E}\Big(\int_0^{t\wedge\tau_k}(t\wedge\tau_k - s)^{-\alpha}ds\int_0^{t\wedge\tau_k}(t\wedge\tau_k - s)^{-\alpha}|f(\hat{V}_k^N(s))|^2ds\Big)$$
$$+ \mathbb{E}\Big(\int_0^{t\wedge\tau_k}(t\wedge\tau_k - s)^{-2\beta}|g(\hat{V}_k^N(s))|^2ds\Big)$$
$$\le C\Big(1+\int_0^{t\wedge\tau_k}(t\wedge\tau_k - s)^{-\alpha}\mathbb{E}\big(|\check{V}_k^N(s)|^2\big)ds + \int_0^{t\wedge\tau_k}(t\wedge\tau_k - s)^{-\alpha}\mathbb{E}\big(|\hat{V}_k^N(s)|^2\big)ds$$
$$+\int_0^{t\wedge\tau_k}(t\wedge\tau_k - s)^{-2\beta}\mathbb{E}\big(|\hat{V}_k^N(s)|^2\big)ds\Big), \tag{3.5}$$

where the positive constant C is independent of N and k. For the sake of simplicity, one can claim that there is a pair of positive constants (α,β) such that $\alpha+\beta<1$ and

$$\sup_{0\le r\le t}\mathbb{E}\big(|V_k^N(r)|^2\big) \le C\Big(1+\sup_{0\le r\le t}\int_0^{r\wedge\tau_k}(t\wedge\tau_k - s)^{-\alpha}\sup_{0\le\eta\le s}\mathbb{E}\big(|V_k^N(\eta)|^2\big)ds\Big)$$
$$= C\Big(1+\sup_{0\le r\le t}(r\wedge\tau_k)^{1-\alpha}\int_0^1(1-u)^{-\alpha}u^{-\beta}\sup_{0\le\eta\le(r\wedge\tau_k)u}\mathbb{E}\big(|V_k^N(\eta)|^2\big)du\Big)$$
$$\le C\Big(1+t^{1-(\alpha+\beta)}\int_0^1(1-u)^{-\alpha}u^{-\beta}\sup_{0\le\eta\le tu}\mathbb{E}\big(|V_k^N(\eta)|^2\big)du\Big)$$
$$= C\Big(1+\int_0^t(t-s)^{-\alpha}\sup_{0\le\eta\le s}\mathbb{E}\big(|V_k^N(\eta)|^2\big)ds\Big). \tag{3.6}$$

Finally, by letting $k\to+\infty$ with Fatou's lemma, complete the proof. □

Lemma 3.2. *Under the assumptions of Lemma 3.1, there exists a positive constant C independent of h such that for all $t\in[t_n,t_{n+1})$, we have:*

$$\mathbb{E}\big(|V^N(t)-\hat{V}^N(t)|^2\big)\le Ch^{\min\{2-2\alpha,1-2\beta\}}. \tag{3.7}$$

Proof. By using the power mean inequality, Itô isometry and Lemma 3.1 as well as Cauchy-Schwarz's inequality that:

$$\mathbb{E}\big(|V^N(t)-\hat{V}^N(t)|^2\big)\le 6\theta^2\mathbb{E}\Big(\int_0^{t_n}|(t-s)^{-\alpha}-(t_n-s)^{-\alpha}|ds\int_0^{t_n}|(t-s)^{-\alpha}-(t_n-s)^{-\alpha}||f(\check{V}^N(s))|^2ds\Big)$$
$$+6\theta^2\mathbb{E}\Big(\int_{t_n}^t(t-s)^{-\alpha}ds\int_{t_n}^t(t-s)^{-\alpha}|f(\check{V}^N(s))|^2ds\Big)$$
$$+6(1-\theta)^2\mathbb{E}\Big(\int_0^{t_n}|(t-s)^{-\alpha}-(t_n-s)^{-\alpha}|ds\int_0^{t_n}|(t-s)^{-\alpha}-(t_n-s)^{-\alpha}||f(\hat{V}^N(s))|^2ds\Big)$$
$$+6(1-\theta)^2\mathbb{E}\Big(\int_{t_n}^t(t-s)^{-\alpha}ds\int_{t_n}^t(t-s)^{-\alpha}|f(\hat{V}^N(s))|^2ds\Big)$$
$$+6\mathbb{E}\Big(\int_0^{t_n}\big((t-s)^{-\beta}-(t_n-s)^{-\beta}\big)^2|g(\hat{V}^N(s))|^2ds\Big)+6\mathbb{E}\Big(\int_{t_n}^t(t-s)^{-2\beta}|g(\hat{V}^N(s))|^2ds\Big), \tag{3.8}$$

where

$$\int_0^{t_n}|(t-s)^{-\alpha}-(t_n-s)^{-\alpha}|ds=\int_0^{t_n}\big((t_n-s)^{-\alpha}-(t-s)^{-\alpha}\big)ds$$
$$=\int_0^{t_n}(t_n-s)^{-\alpha}ds-\int_0^t(t-s)^{-\alpha}ds+\int_{t_n}^t(t-s)^{-\alpha}ds$$
$$=\frac{1}{(t_n-t)^{1-\alpha}}\Big(t_n^{1-\alpha}-t^{1-\alpha}\Big)+\int_{t_n}^t(t-s)^{-\alpha}ds$$

$$\leq \int_{t_n}^{t}(t-s)^{-\alpha}ds \leq t_n^{-\alpha}\int_{t_n}^{t}(t-s)^{-\alpha}ds \leq Ch^{1-\alpha}, \tag{3.9}$$

and

$$\begin{aligned}\int_0^{t_n}\Big((t-s)^{-\beta}-(t_n-s)^{-\beta}\Big)^2 ds &= \int_0^{t_n}\big((t-s)^{-2\beta}+(t_n-s)^{-2\beta}-2(t-s)^{-\beta}(t_n-s)^{-\beta}\big)ds\\
&\leq \int_0^{t_n}\big((t-s)^{-2\beta}+(t_n-s)^{-2\beta}-2(t-s)^{-\beta}(t-s)^{-\beta}\big)ds\\
&= \int_0^{t_n}(t_n-s)^{-2\beta}-(t-s)^{-2\beta}ds\\
&= \int_0^{t_n}(t_n-s)^{-2\beta}ds-\int_0^{t}(t-s)^{-2\beta}ds+\int_{t_n}^{t}(t-s)^{-2\beta}ds\\
&= \frac{1}{1-2\beta}\Big(t_n^{1-2\beta}-t^{1-2\beta}\Big)+\int_{t_n}^{t}(t-s)^{-2\beta}ds \leq Ch^{1-2\beta},\end{aligned} \tag{3.10}$$

which joined with the linear growth condition (2.2) and relations (3.8)-(3.10) can exhibit:

$$\begin{aligned}\mathbb{E}\big(|V^N(t)-\hat{V}^N(t)|^2\big) &\leq 6\theta^2 CK(1+\mathbb{E}\big(|\check{V}^N(s)|^2\big)h^{2-2\alpha}+6(1-\theta)^2CK(1+\mathbb{E}\big(|\hat{V}^N(s)|^2\big)h^{2-2\alpha}\\
&+6CK(1+\mathbb{E}\big(|\hat{V}^N(s)|^2\big)h^{1-2\beta} \leq Ch^{\min\{2-2\alpha,1-2\beta\}}.\end{aligned} \tag{3.11}$$

Finally, using the fact $h = T/N$ completes the proof. □

Now, we continue to introduce a modified version of the stochastic θ-approximation to avoid simulating extra stochastic integrals in the actual numerical implementation. To avoid disposing of the singular point of kernel functions, the stochastic θ-scheme can be introduced as:

$$\mathcal{V}_n = \mathcal{V}_0+\theta\sum_{i=0}^{n-1}(t_n-t_i)^{-\alpha}f(\mathcal{V}_{i+1})\Delta+(1-\theta)\sum_{i=0}^{n-1}(t_n-t_i)^{-\alpha}f(\mathcal{V}_i)\Delta+\sum_{i=0}^{n-1}(t_n-t_i)^{-\beta}g(\mathcal{V}_i)\Delta W_i, \tag{3.12}$$

where $n = 1, 2, \cdots, N$, $\Delta := t_{i+1}-t_i$ and $\Delta W_i := W(t_{i+1})-W(t_i)$ $(i = 0, 1, \cdots, N-1)$ denote the increments of Brownian motion. For its convergence analysis, we also introduce a continuous-time version as follows:

$$\mathcal{V}(t) = \mathcal{V}_0+\theta\int_0^t(t-\underline{s})^{-\alpha}f(\check{\mathcal{V}}(s))ds+(1-\theta)\int_0^t(t-\underline{s})^{-\alpha}f(\hat{\mathcal{V}}(s))ds+\int_0^t(t-\underline{s})^{-\beta}g(\hat{\mathcal{V}}(s))dW(s), \tag{3.13}$$

where $\underline{s} := t_i$ for $s \in [t_i, t_{i+1})$ denotes the left endpoint, $\check{s} := t_{i+1}$ for $s \in (t_i, t_{i+1}]$ denotes the right endpoint, $\check{\mathcal{V}}(s) := \mathcal{V}_{i+1}$ for $s \in (t_i, t_{i+1}]$ and $\hat{\mathcal{V}}(s) := X_i$ for $s \in [t_i, t_{i+1})$ stands for the piecewise constant interpolation of the stochastic θ-scheme (3.12). Note that $\mathcal{V}(t_{n+1}) = \mathcal{V}_{n+1}$ and $\mathcal{V}(t_n) = \mathcal{V}_n$.

3.3. Convergence rate.

Lemma 3.3. *If the positive constant α and satisfies $\alpha \in (0, 1)$, then there exists a positive constant C independent of h such that:*

$$\int_0^t |(t-s)^{-\alpha}-(t-\underline{s})^{-\alpha}|ds \leq Ch^{1-\alpha}. \tag{3.14}$$

Lemma 3.4. *If the positive constant β satisfies $\beta \in (0, \frac{1}{2})$, then there exists a positive constant C independent of h such that for all $t \in [t_n, t_{n+1}]$, $n = 1, 2, \cdots, N-1$:*

$$\int_0^{t_n} |(t-\underline{s})^{-\beta}-(t_n-\underline{s})^{-\beta}|^2 ds \leq Ch^{1-2\beta}, \tag{3.15}$$

and for any $t \in [0, T]$:

$$\int_0^t |(t-s)^{-\beta}-(t-\underline{s})^{-\beta}|^2 ds \leq Ch^{1-2\beta}. \tag{3.16}$$

Lemma 3.5. *([12]) Under the assumptions of Lemma 3.1, there exists a positive constant C independent of h such that:*

$$\sup_{0\le t\le T}\mathbb{E}\big(|\check{\mathcal{V}}(t)|^2\big)\vee\sup_{0\le t\le T}\mathbb{E}\big(|\hat{\mathcal{V}}(t)|^2\big)\le\sup_{0\le t\le T}\mathbb{E}\big(|\mathcal{V}(t)|^2\big)\le C,\qquad \textit{for given } T>0. \tag{3.17}$$

Lemma 3.6. *Under the assumptions of Lemma 3.1, there exists a positive constant C independent of h such that for all $t\in[0,T]$:*

$$\mathbb{E}\big(|\mathcal{V}(t)-\check{\mathcal{V}}(t)|^2\big)\vee\mathbb{E}\big(|\mathcal{V}(t)-\hat{\mathcal{V}}(t)|^2\big)\le Ch^{\min\{2-2\alpha,1-2\beta\}}. \tag{3.18}$$

Based on Lemma 3.3, the proofs of Lemmas 3.5 and 3.6 are similar with the proofs of Lemmas 3.1 and 3.2, respectively. Now, we prove the mean square convergence of the proposed method, which relaxes the integrable limitations of singular kernels.

Theorem 3.7. *Under the assumptions of Lemma 3.2, there exists a positive constant C independent of h, such that for all $t\in[0,T]$ we have:*

$$\Big(\mathbb{E}\big(|V(t)-\mathcal{V}(t)|^2\big)\Big)^{\frac{1}{2}}\le Ch^{\min\{1-\alpha,\frac{1}{2}-\beta\}}. \tag{3.19}$$

Proof. By adding some auxiliary terms, it follows from the SHM (1.1) and the stochastic θ-scheme (3.13) that:

$$\begin{aligned}\mathbb{E}\big(|V(t)-\mathcal{V}(t)|^2\big)\le{}&6\theta^2\mathbb{E}\Big(\int_0^t(t-s)^{-\alpha}\big(f(V(s))-f(\check{\mathcal{V}}(s)\big)ds\Big)^2\\&+6\theta^2\mathbb{E}\Big(\int_0^t\big((t-s)^{-\alpha}-(t-\underline{s})^{-\alpha}\big)f(\check{\mathcal{V}}(s))ds\Big)^2\\&+6(1-\theta)^2\mathbb{E}\Big(\int_0^t(t-s)^{-\alpha}\big(f(V(s))-f(\hat{\mathcal{V}}(s)\big)ds\Big)^2\\&+6(1-\theta)^2\mathbb{E}\Big(\int_0^t\big((t-s)^{-\alpha}-(t-\underline{s})^{-\alpha}\big)f(\hat{\mathcal{V}}(s))ds\Big)^2\\&+6\mathbb{E}\Big(\int_0^t(t-s)^{-\beta}\big(g(V(s))-g(\hat{\mathcal{V}}(s)\big)dW(s)\Big)^2\\&+6\mathbb{E}\Big(\int_0^t\big((t-s)^{-\beta}-(t-\underline{s})^{-\beta}\big)g(\hat{\mathcal{V}}(s))dW(s)\Big)^2\\={}&J_1+J_2+J_3+J_4+J_5+J_6.\end{aligned} \tag{3.20}$$

Now by using Cauchy-Schwarz's inequality for J_1-J_4, Itô isometry for J_5 and J_6 and linear growth condition, we can obtain:

$$\begin{aligned}J_1&\le6\theta^2\mathbb{E}\Big(\int_0^t(t-s)^{-\alpha}ds\int_0^t(t-s)^{-\alpha}\big(f(V(s))-f(\check{\mathcal{V}}(s)\big)^2ds\Big)\\&\le6L\theta^2\frac{1}{1-\alpha}t^{1-\alpha}\int_0^t(t-s)^{-\alpha}\Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2+\mathbb{E}|\mathcal{V}(s)-\check{\mathcal{V}}(s)|^2\Big)ds\\&=C_1\int_0^t(t-s)^{-\alpha}\Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2+\mathbb{E}|\mathcal{V}(s)-\check{\mathcal{V}}(s)|^2\Big)ds,\end{aligned} \tag{3.21}$$

$$\begin{aligned}J_2&\le6\theta^2\mathbb{E}\Big(\int_0^t|(t-s)^{-\alpha}-(t-\underline{s})^{-\alpha}|ds\int_0^t|(t-s)^{-\alpha}-(t-\underline{s})^{-\alpha}||f(\check{\mathcal{V}}(s)|^2ds\Big)\\&\le6K\theta^2\big(1+\mathbb{E}|\check{\mathcal{V}}(s)|^2\big)h^{2-2\alpha}=C_2h^{2-2\alpha},\end{aligned} \tag{3.22}$$

$$J_3\le6(1-\theta)^2\mathbb{E}\Big(\int_0^t(t-s)^{-\alpha}ds\int_0^t(t-s)^{-\alpha}\big(f(V(s))-f(\hat{\mathcal{V}}(s)\big)^2ds\Big)$$

$$\le 6L(1-\theta)^2 \frac{1}{1-\alpha} t^{1-\alpha} \int_0^t (t-s)^{-\alpha} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\hat{\mathcal{V}}(s)|^2 \Big) ds$$
$$= C_3 \int_0^t (t-s)^{-\alpha} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\hat{\mathcal{V}}(s)|^2 \Big) ds, \tag{3.23}$$

$$J_4 \le 6(1-\theta)^2 \mathbb{E}\Big(\int_0^t |(t-s)^{-\alpha} - (t-\underline{s})^{-\alpha}| ds \int_0^t |(t-s)^{-\alpha} - (t-\underline{s})^{-\alpha}||f(\hat{\mathcal{V}}(s)|^2 ds \Big)$$
$$\le 6K(1-\theta)^2 \big(1+\mathbb{E}|\check{\mathcal{V}}(s)|^2\big) h^{2-2\alpha} = C_4 h^{2-2\alpha}, \tag{3.24}$$

$$J_5 \le 6\mathbb{E}\Big(\int_0^t (t-s)^{-2\beta} \big(g(x(s)) - g(\hat{\mathcal{V}}(s)\big)^2 ds \Big)$$
$$\le 6L \int_0^t (t-s)^{-2\beta} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\hat{\mathcal{V}}(s)|^2 \Big) ds$$
$$= C_5 \int_0^t (t-s)^{-2\beta} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\hat{\mathcal{V}}(s)|^2 \Big) ds, \tag{3.25}$$

$$J_6 \le 6\mathbb{E}\Big(\int_0^t \big((t-s)^{-\beta} - (t-\underline{s})^{-\beta}\big)^2 |g(\hat{\mathcal{V}}(s))|^2 ds \Big) \le 6K\big(1+\mathbb{E}|\check{\mathcal{V}}(s)|^2\big) h^{1-2\beta} = C_6 h^{1-2\beta}, \tag{3.26}$$

then we have:

$$\mathbb{E}\big(|V(t)-\mathcal{V}(t)|^2\big) \le C\Big\{ \int_0^t (t-s)^{-\alpha} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\check{\mathcal{V}}(s)|^2 \Big) ds$$
$$+ \int_0^t (t-s)^{-\alpha} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\hat{\mathcal{V}}(s)|^2 \Big) ds$$
$$+ \int_0^t (t-s)^{-2\beta} \Big(\mathbb{E}|V(s)-\mathcal{V}(s)|^2 + \mathbb{E}|\mathcal{V}(s)-\hat{\mathcal{V}}(s)|^2 \Big) ds + h^{2-2\alpha} + h^{1-2\beta} \Big\}, \tag{3.27}$$

which with Lemma 3.6 we can obtain:

$$\mathbb{E}\big(|V(t)-\mathcal{V}(t)|^2\big) \le C\Big(\int_0^t (t-s)^{-\alpha} \mathbb{E}\big(|V(s)-\mathcal{V}(s)|^2\big) ds + h^{\min\{2-2\alpha, 1-2\beta\}} \Big). \tag{3.28}$$

Then, it holds by a similar deducing process for the estimate (3.6) that:

$$\sup_{0\le r\le t} \mathbb{E}\big(|V(r)-\mathcal{V}(r)|^2\big) \le C\Big(\int_0^t (t-s)^{-\alpha} \sup_{0\le r\le s} \mathbb{E}\big(|V(r)-\mathcal{V}(r)|^2\big) ds + h^{\min\{2-2\alpha, 1-2\beta\}} \Big). \tag{3.29}$$

Finally, using weakly singular Gronwall's inequality and arbitrariness of $t \in [0, T]$ completes the proof. □

4. Numerical illustrations

This section is devoted to presenting an example to illustrate our numerical results for the stochastic θ-scheme (3.13). In this section, we will implement the proposed method to investigate its actual results for the given example and to verify the theoretical results in section 3. In a similar way to [11], we use the sample average to approximate the expectation. More precisely, we measure the mean square error of numerical solutions at the terminal time $t_N = T = 1$ by:

$$e_{h,T}^{strong} := \Big(\frac{1}{M} \sum_{i=1}^{M} |V^{(i)}(t_N) - \mathcal{V}^{(i)}(t_N)|^2 \Big)^{\frac{1}{2}}, \tag{4.1}$$

where $V^{(i)}(t_N)$ and $\mathcal{V}^{(i)}(t_N)$ are the exact solution and the numerical solution, respectively, in the ith sample path. In all error estimations, we use the numerical solution of the stochastic θ-scheme with step size $h^* = 2^{-14}$ as a good approximation of the exact solution. We also compute numerical solutions using step sizes $h = 16h^*, 32h^*, 64h^*$ and

TABLE 1. Convergence orders and CPU time of the numerical solution with $\epsilon = 10^{-12}$ for Eq. (1.1).

h	$\theta = 0.2$	$\theta = 0.5$	$\theta = 0.8$	CPU time
2^{-7}	0.2390	0.2496	0.2783	1.08
2^{-8}	0.1639	0.1804	0.2026	3.69
2^{-9}	0.1168	0.1329	0.1476	13.56
2^{-10}	0.0868	0.1008	0.1155	26.20
order	0.0824	0.0989	0.1105	59.43

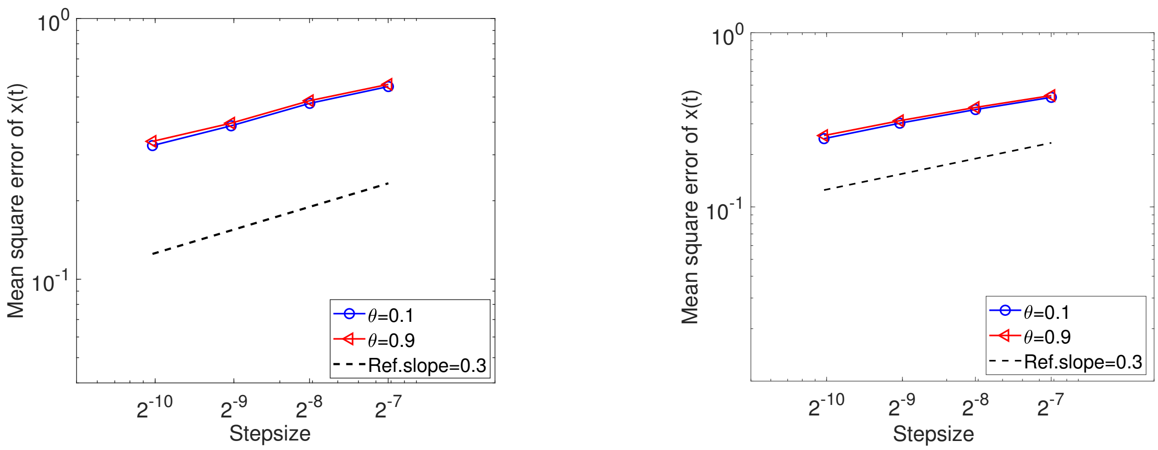

FIGURE 1. Mean square errors of numerical solution for Eq. (4.2)(left: case I; right: case II).

TABLE 2. European call option prices with $T = 1$, $r = 0.3$, $h = 2^{-10}$ and 10^4 number of paths.

K	$\alpha = \beta = 0.6$ (*M.C. %95 simulation*)		$\alpha = \beta = 0.75$ (*M.C. %95 simulation*)		$\alpha = \beta = 0.9$ (*M.C. %95 simulation*)	
80	11.7631	$(11.7411, 11.7831)$	12.1617	$(12.1322, 12.2073)$	12.5093	$(12.4831, 12.5317)$
100	7.4941	$(7.4115, 7.5253)$	7.7640	$(7.7323, 7.8061)$	8.1814	$(8.1392, 8.2163)$
120	3.3088	$(3.2711, 3.3241)$	3.8115	$(3.7846, 3.8359)$	4.2410	$(4.2175, 4.2806)$

$128h^*$ on the same Brownian path and obtain the corresponding errors. We generate $M = 5000$ different discretized Brownian paths over $[0, T]$ and calculate the mean square errors at the endpoint T. To visualize the experimental results, we plot error $e^{strong}_{h,T}$ against h in log-log scale.

Now, we consider the following stochastic rough Heston model with one singular point:

$$\begin{cases} dS_t = S_t\sqrt{V_t}dW_t, \\ V_t = V_0 + \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^t (t-s)^{\alpha-1}\big(\omega(s) - \lambda V_s\big)ds + \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^t (t-s)^{\beta-1}\nu\sqrt{V_s}dB_s, \quad t \in [0,1]. \end{cases} \tag{4.2}$$

where the positive arguments α and β take the following two cases:

- Case I: $\alpha = 0.4$ and $\beta = 0.1$,
- Case II: $\alpha = 0.6$ and $\beta = 0.2$.

The drift and diffusion functions satisfy the assumptions of Theorems 3.7. To verify the convergence rates, we choose $V_0 = 0.04$, $\rho = -0.4$, $\lambda = \nu = 0.2$, $\alpha = 0.5$, $\beta = 0.8$ and $\omega = 0.03$ for the proposed scheme. The numerical results are displayed in Tables 1 and 2.

Monte Carlo (M.C.) simulations are a powerful tool for solving SDEs, especially when dealing with singularities or complex systems. M.C. used to generate correlated random variables by the relation (1.2). We generate a large number of random samples 10^4 simulations, to approximate the solution of the SDE, which typically includes computing the average of the results from the random paths to approximate the solution [6, 26].

5. Conclusion

In this paper, we examine the existence and uniqueness of the numerical solution for the stochastic Heston model with a singular point under the global Lipschitz condition and the linear growth condition. In addition, we succeed in extracting the convergence ratio by the stochastic calculus, which depends on the values of α and β. We tabulate some numerical results for the convergence analysis for different step size values and several theta values. Also, we could evaluate their correspondence option values with different kinds of strike prices. The results show that our implementation is rough for the proposed problem.

Acknowledgment

The fourth author is supported by a research postdoctoral grant of the University of Tabriz (No. 940).

References

[1] N. Agram and B. Oksendal, *Malliavin calculus and optimal control of stochastic Volterra equations*, J. Optim. Theory Appl., *167* (2015), 1070–1094.
[2] D. Ahmadian and O. Farkhondeh Rouz, *Analysis on mean-square and almost sure exponential stability of numerical method for stochastic differential equations with constant delay*, Int. J. Appl. Math. Stat., *56* (2017), 1–13.
[3] D. Ahmadian, O. Farkhondeh Rouz, and L. V. Ballestra, *Stability analysis of split-step θ-Milstein method for a class of n-dimensional stochastic differential equations*, Appl. Math. Comput., *348* (2019), 413–424.
[4] D. Ahmadian and O. Farkhondeh Rouz, *Exponential mean-square stability of numerical solutions for stochastic delay integro-differential equations with Poisson jump*, J. Inequalities Appl., *186* (2020), 1–33.
[5] D. Ahmadian and O. Farkhondeh Rouz, *Boundedness and convergence analysis of stochastic differential equations with Hurst brownian motion*, Bol. Soc. Paran. Mat., *38* (2020), 187–196.
[6] D. Ahmadian, L. V. Ballestra, and F. Shokrollahi, *A Monte-Carlo approach for pricing arithmetic Asian rainbow options under the mixed fractional Brownian motion*, Chaos Solit. Fractals., *158* (2022), 112023.
[7] D. Ahmadian and O. Farkhondeh Rouz, *Mean-square stability of a constructed Third-order stochastic Runge–Kutta schemes for general stochastic differential equations*, Comput. Methods Differ. Equ., *10* (2022), 617–638.
[8] M. Berger and V. Mizel, *Volterra equations with Itô integrals-I*, J. Int. Equ., *2* (1980), 187–245.
[9] M. Berger and V. Mizel, *Volterra equations with Itô integrals II*, J. Int. Equ., *2* (1980), 319–337.
[10] D. Conte, R. D'Ambrosio, and B. Paternoster, *Improved θ-methods for stochastic Volterra integral equations*, Commun. Nonlinear Sci. Numer. Simulat., *93* (2021), 105528.
[11] X. Dai and A. Xiao, *Levy-driven stochastic Volterra integral equations with doubly singular kernels: Existence, uniqueness and a fast EM method*, Adv. Comput. Math., *46* (2020), 1–23.
[12] X. Dai, A. Xiao, and W. Bu, *Stochastic fractional integro-differential equations with weakly singular kernels: Well-posedness and Euler–Maruyama approximation*, Discrete Contin. Dyn. Syst. Ser. B., *27* (2022), 4231–4253.
[13] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of caputo type*, Springer, (2010).
[14] O. E. Euch and M. Rosenbaum, *Perfect hedging in rough Heston models*, Ann. Appl. Probab., *28* (2018), 3813–3856.
[15] O. Farkhondeh Rouz, D. Ahmadian, and M. Milev, *Exponential mean-square stability of two classes of theta Milstein methods for stochastic delay differential equations*, AIP Conference Proceedings, *1910* (2017), 060015-1–060015-14.
[16] O. Farkhondeh Rouz, D. Ahmadian, A. Jodayree Akbarfam, and M. Milev, *A note on the almost sure exponential stability of the Milstein method for stochastic delay differential equations with Jumps*, Int. J. Pure. Appl. Math., *1* (2017), 201–216.
[17] O. Farkhondeh Rouz and D. Ahmadian, *Stability analysis of two classes of improved backward Euler methods for stochastic delay differential equations of neutral type*, Comput. Methods Differ. Equ., *5* (2017), 201–213.
[18] O. Farkhondeh Rouz, *Preserving asymptotic mean-square stability of stochastic theta scheme for systems of stochastic delay differential equations*, Comput. Methods Differ. Equ., *8* (2020), 468–479.

[19] P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Springer, (1992).
[20] M. Li, Ch. Huang, and Y. Hu, *Asymptotic separation for stochastic Volterra integral equations with doubly singular kernels*, Appl. Math. Letters., *113* (2021), 106880.
[21] Y. Li, W. Song, Y. Jiang, and A. Kudreyko, *Galerkin approximation for stochastic Volterra integral equations with doubly singular kernels*, Fractal Fract., *6* (2022), 1–12.
[22] H. Liang, Z. Yang, and J. Gao, *Strong superconvergence of the Euler-Maruyama method for linear stochastic Volterra integral equations*, J. Comput. Appl. Math., *317* (2017), 447–457.
[23] X. Mao, *Stochastic differential equations and applications*, Horwood Publishing, Chichester, (2008).
[24] A. Richard, X. Tan, and F. Yang, *On the discrete-time simulation of the rough Heston model*, SIAM J. Financial Math., *14* (2023), 223–249.
[25] A. Safdari Vaighani, D. Ahmadian, and R. Javid Jahromi, *An approximation scheme for option pricing under Two-state continuous CAPM*, Comput. Econ., *57* (2021), 1373–1385.
[26] F. Shokrollahi, D. Ahmadian, and L. V. Ballestra, *Pricing Asian options under the mixed fractional Brownian motion with jumps*, Math. Comput. Simul., *226* (2024), 172–183.
[27] D. Szynal and S. Wedrychowicz, *On solutions of a stochastic integral equation of the Volterra type with applications for chemotherapy*, J. Appl. Probab., *25* (1988), 257–267.
[28] S. Vahdati, *A wavelet method for stochastic Volterra integral equations and its application to general stock model*, Comput. Methods Differ. Equ., *5* (2017), 170–188.
[29] Z. Wang, *Existence and uniqueness of solutions to stochastic Volterra equations with singular kernels and non-Lipschitz coefficients*, Statist. Probab. Lett., *78* (2008), 1062–1071.
[30] A. Wazwaz and R. Rach, *Two reliable methods for solving the Volterra integral equation with a weakly singular kernel*, J. Comput. Appl. Math., *302* (2016), 71–80.
[31] Y. Xiao, J. Shi, and Z. Yang, *Split-step collocation methods for stochastic Volterra integral equations*, J. Integral Equ. Appl., *30* (2018), 197–218.
[32] A. Xiao, X. Dai, and W. Bu, *Well-posedness and EM approximation for nonlinear stochastic fractional integro-differential equations with weakly singular kernels*, J. Comput. Appl. Math., *356* (2019), 377–390.
[33] X. Zhang, *Euler schemes and large deviations for stochastic Volterra equations with singular kernels*, J. Differ. Equ., *244* (2008), 2226–2250.
[34] Q. Zhao, R. Wang, and R. Wei, *Exponential utility maximization for an insurer with time-inconsistent preferences*, Insur., *70* (2016), 89–104.