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An effective Legendre wavelet technique for the time-fractional Fisher equation

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Abstract

This study modifies the time-fractional Fisher equation by adding a source term and generalizing the non-linear power to an arbitrary order. A numerical technique is proposed for the modified time-fractional Fisher equation using Legendre wavelets and the quasilinearization technique. The non-linear term is iteratively linearized using the quasilinearization technique. The convergence analysis and error estimates of the proposed method are studied. Several test problems are solved using the proposed technique, and numerical outcomes are contrasted with those obtained using some other approaches existing in the literature.

Keywords. Numerical methods, Fisher's equation, Legendre wavelets, Quasilinearization.
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1. Introduction

The history of fractional calculus goes back to Leibniz. Despite its long history, the theory and applications of fractional calculus did not advance until the 20th century. Recently, numerous scientists and engineers have been interested in fractional differential equations because fractional differential equations seem to be more convenient models for many applications in viscoelasticity theory, fluid dynamics, electrochemistry, biology, etc. [9, 11, 24]. Furthermore, there are mathematical models that can only be described using fractional differential equations. For example, the Bagley-Torvik equation is used to describe how a rigid plate moves when submerged in a Newtonian fluid, and it involves a fractional order differential term [32]. However, it is not possible to find an exact solution for the majority of fractional differential equations. Therefore, we are interested in numerical methods for fractional differential equations.

For numerical solution of fractional differential equations, there exist many different methods such as the variational iteration method (VIM) [21], the homotopy perturbation method (HPM) [22], and the Adomian decomposition method (ADM) [27]. In addition to these methods, spectral methods, such as collocation [31], Galerkin [25], and tau [5] methods are effective to solve fractional differential equations due to their fast convergence properties.

Over the last three decades, wavelet theory has been developed and applied to many areas of research such as signal processing [18], image processing [23, 35], inverse problems [18], etc. Appropriately scaled and shifted wavelets can be used to form a Schauder basis for the space of square-integrable functions on the interval $[a, b]$. Hence, any function $f \in L^2[a, b]$ can be written as an infinite sum of wavelets. Using this feature, many researchers have proposed computational techniques for the numerical solution of fractional differential equations [2, 7, 8, 12, 16, 19, 26, 29, 33]. With the aid of this feature, we are going to propose a numerical method using the Legendre wavelet collocation method and quasilinearization technique.

In this article, we consider the following form of the time-fractional Fisher equation

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$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \delta \frac{\partial^2 y(x,t)}{\partial x^2} + \lambda y(x,t)(1 - y^p(x,t)) + q(x,t), \qquad 0 \le x \le 1, \quad 0 \le t \le 1. \tag{1.1}$$

The initial condition is

$$y(x,0) = w(x), \tag{1.2}$$

and boundary conditions are

$$y(0,t) = z_1(t), \tag{1.3}$$

$$y(1,t) = z_2(t), \tag{1.4}$$

where $\delta, \lambda, p \in \mathbb{R}$; $w(x)$, $z_1(t)$, $z_2(t)$, and $q(x,t)$ are known functions. $\dfrac{\partial^\alpha y(x,t)}{\partial t^\alpha}$ denotes the α-th order Caputo fractional derivative of $y = y(x,t)$ with respect to t, where $0 < \alpha \le 1$. In 1937, Fisher proposed the following simplified form of Eq. (1.1) as a model for the spread of a new mutant gene in a population, where y represents the density of the mutant gene in the population [10]:

$$\frac{\partial y(x,t)}{\partial t} = \delta \frac{\partial^2 y(x,t)}{\partial x^2} + \lambda y(x,t)(1 - y(x,t)). \tag{1.5}$$

The Fisher equation is also used in many branches of science and technology, such as chemical kinetics [28], branching Brownian motion [6], epidemics and bacteria [15].

Aghazadeh *et al.* [1] explained how to solve nonlinear fractional partial differential equations (including the time-fractional Fisher equation) using the Laguerre wavelet and the Adomian decomposition method. Ahmadnezhad *et al.* [3] used the Haar wavelet method together with Picard iteration to solve the time-fractional Fisher equation numerically. Mohyud-Din and Noor [20] obtained numerical results for the Fisher equation using the modified variational iteration method. Wazwaz and Gorguis [34] studied the Adomian decomposition method to find solutions to the Fisher equation. Seçer and Çınar [30] developed a computational approach for the time-fractional Fisher equation based on Jacobi wavelets.

2. Preliminaries

The main goal of this section is to present the fundamentals of fractional calculus, Legendre wavelets, and quasi-linearization technique.

2.1. Fractional Integral and Derivative.

Definition 2.1. *[14] Let $\alpha \in \mathbb{R}^+$ and $\Omega = [a,b]$ be a finite interval, where $-\infty < a < b < \infty$. For a function $f \in L^1[a,b]$, its Riemann-Liouville integral of order α is defined by*

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx, \tag{2.1}$$

where Γ represents the gamma function and $t \in [a,b]$. For $\alpha = 0$, we set

$${}_aI_t^0 f(t) = f(t). \tag{2.2}$$

We have the following useful properties for the Riemann-Liouville fractional integral [14]:

(1) Consider the function $f(t) = (t-a)^\gamma$ with $\gamma > -1$. For $\alpha \in \mathbb{R}^+$, we have

$${}_aI_t^\alpha f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}. \tag{2.3}$$

(2) Let f be an absolutely integrable function over the interval $[a,b]$, that is $f \in L^1[a,b]$. Suppose that $\alpha, \beta \in \mathbb{R}^+$. Then there holds

$${}_aI_t^\alpha\, {}_aI_t^\beta f(t) = {}_aI_t^\beta\, {}_aI_t^\alpha f(t) = {}_aI_t^{\alpha+\beta} f(t). \tag{2.4}$$

Definition 2.2. *[14] Let $\alpha \in \mathbb{R}^+$ and $\Omega = [a, b]$ be a finite interval, where $-\infty < a < b < \infty$. For a function $f \in L^1[a, b]$, its Caputo derivative of order α is defined by*

$$ {}_a^C D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)\, dx, & n-1 < \alpha < n,\ n \in \mathbb{N} \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n,\ n \in \mathbb{N} \end{cases} \tag{2.5} $$

where Γ represents the gamma function and $t \in [a, b]$.

We have the following useful properties for the Riemann-Liouville fractional integral and the Caputo fractional derivative [14]:

(1) Let $\alpha \in \mathbb{R}^+$ such that $n - 1 < \alpha < n$, where $n \in \mathbb{N}$. Suppose that $f \in C[a, b]$. Then

$$ {}_a^C D_t^\alpha\, {}_a I_t^\alpha f(t) = f(t) \tag{2.6} $$

for all $t \in [a, b]$.

(2) Let $\alpha \in \mathbb{R}^+$ such that $n - 1 < \alpha < n$, where $n \in \mathbb{N}$. If $f \in C^n[a, b]$, then

$$ {}_a I_t^\alpha\, {}_a^C D_t^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(t-a)^k}{k!}, \quad \text{for all} t \in [a, b]. \tag{2.7} $$

2.2. **Wavelets.** Wavelets are a special family of functions. The family of functions is created by scaling and shifting a single function, known as the mother wavelet. Each wavelet function in the family can be expressed in terms of the mother wavelet, the scaling parameter p, and the shifting parameter r as follows:

$$ \psi_{p,r}(t) = |p|^{-1/2} \psi\left(\frac{t-r}{p}\right), \quad p, r \in \mathbb{R}, \quad p \neq 0. \tag{2.8} $$

By restricting the scaling parameter p and shifting parameter r to $p = p_0^{-k}$ and $r = n r_0 p_0^{-k}$, where $p_0 > 1$, $r_0 > 0$, and $k, n \in \mathbb{Z}$; we get a countable family of wavelets Ψ whose members are of the form

$$ \psi_{k,n}(t) = {p_0}^{k/2} \psi({p_0}^k t - n r_0). \tag{2.9} $$

The countable family of wavelets $\Psi = \{\psi_{k,n} : k, n \in \mathbb{Z}\}$ forms a Schauder basis for the Hilbert space $L^2(\mathbb{R})$.

2.2.1. *Legendre Polynomials and Legendre Wavelets.* The Legendre polynomials are a special family of orthogonal polynomials on the interval $[-1, 1]$. The m-th degree Legendre polynomial can be determined using the recursion formula

$$ L_0(t) = 1, \qquad L_1(t) = t, \tag{2.10} $$

$$ (m+1)L_{m+1}(t) = (2m+1)tL_m(t) - mL_{m-1}(t), \qquad m = 1, 2, 3, \ldots . \tag{2.11} $$

The Legendre polynomials are orthogonal with respect to the weight function $w(t) = 1$ on $[-1, 1]$. More precisely, we have

$$ \langle L_m, L_n \rangle_2 = \int_{-1}^{1} L_m(t) L_n(t)\, dt = \begin{cases} \dfrac{2}{2m+1}, & m = n, \\ 0, & \text{Otherwise}. \end{cases} \tag{2.12} $$

An explicit formula for the Legendre polynomials is given by

$$ L_m(t) = \sum_{d=0}^{m} \binom{m}{d} \binom{m+d}{d} \left(\frac{t-1}{2}\right)^d . \tag{2.13} $$

For practical use of polynomials on the interval $[0, 1]$, the shifted Legendre polynomials on the interval $[0, 1]$ can be expressed as

$$ G_m(t) = L_m(2t - 1), $$

where L_m is the m-th degree Legendre polynomial.

Legendre wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ consist of Legendre polynomials and have four parameters. Legendre wavelets can be described on the interval $[0,1)$ as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\dfrac{2m+1}{2}}2^{k/2}L_m(2^k t - 2n + 1), & \dfrac{2n-2}{2^k} \le t < \dfrac{2n}{2^k}, \\ 0, & \text{Otherwise.} \end{cases} \tag{2.14}$$

Here, k is the resolution level and it can be any positive integer. n can take the values $n = 1, 2, \ldots, 2^{k-1}$. m represents the degree of Legendre polynomials and it can be $m = 0, 1, \ldots, M-1$, where $M \in \mathbb{Z}^+$ is a fixed number. The term $\left(\dfrac{2m+1}{2}\right)^{1/2}$ is required for orthonormality. The scaling parameter is $p = 2^{-k}$ and the shifting parameter is $r = (2n-1)2^{-k}$. L_m denotes the m-th degree Legendre polynomials defined on $[-1,1]$. It should be noted that

$$\langle \psi_{n_1,m_1}, \psi_{n_2,m_2} \rangle_2 = \int_0^1 \psi_{n_1,m_1}(t)\psi_{n_2,m_2}(t)\,dt = \begin{cases} 1, & n_1 = n_2, \text{ and } m_1 = m_2, \\ 0, & \text{Otherwise.} \end{cases} \tag{2.15}$$

Equivalently, Legendre wavelets can be defined using the shifted Legendre polynomials as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\dfrac{2m+1}{2}}2^{k/2}G_m(2^{k-1} t - n + 1), & \dfrac{n-1}{2^{k-1}} \le t < \dfrac{n}{2^{k-1}}, \\ 0, & \text{Otherwise.} \end{cases} \tag{2.16}$$

Using Legendre wavelets, any function $y(x,t) \in L^2([0,1) \times [0,1))$ may be written as follows:

$$y(x,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} a_{n,m,p,q}\psi_{n,m}(x)\psi_{p,q}(t), \tag{2.17}$$

where

$$\begin{aligned} a_{n,m,p,q} &= \langle \psi_{n,m}, \langle y(x,t), \psi_{p,q}(t) \rangle \rangle & (2.18) \\ &= \int_0^1 \int_0^1 y(x,t)\psi_{n,m}(x)\psi_{p,q}(t)\,dx\,dt. & (2.19) \end{aligned}$$

If we truncate the infinite series (2.17), we can approximate $y(x,t)$

$$y(x,t) \approx y_{k,M,h,Q}(x,t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{p=1}^{2^{h-1}}\sum_{q=0}^{Q-1} a_{n,m,p,q}\psi_{n,m}(x)\psi_{p,q}(t), \tag{2.20}$$

where $k, h, M, Q \in \mathbb{Z}^+$. If we change the indices n, m and p, q by $i = M(n-1) + m + 1$ and $j = Q(p-1) + q + 1$, respectively; we can rewrite approximation (2.20) as

$$y(x,t) \approx y_{k,M,h,Q}(x,t) = \sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{h-1}Q} a_{i,j}\psi_i(x)\psi_j(t). \tag{2.21}$$

2.3. **alpha-Order Integral of the Legendre Wavelet.** The α-th order Riemann-Liouville integrals of Legendre wavelets are required to implement the numerical scheme that will be introduced in section 4. The implementation of numerical methods using pen and paper can be tedious and error-prone. Therefore, computers are used to implement numerical methods in the modern world. Although computers can be used to calculate some Riemann-Liouville integrals, it can still take a long time to do so for some of these integrals. Thus, we will present a formula for calculating the Riemann-Liouville integral of Legendre wavelets. In our formula, the Riemann-Liouville integral of Legendre wavelets can be calculated using finite sums. And this makes the calculation of the Riemann-Liouville integral of Legendre wavelets on a computer can be considerably faster.

Theorem 2.3. *For any Legendre wavelet $\psi_{n,m}(t)$ on the interval $[0,1)$, its Riemann-Liouville integral of order α can be calculated by*

$$
{}_0I_t^{\alpha}\psi_{n,m}(t)=\begin{cases}
0, & \text{if} \quad t<\dfrac{2n-2}{2^k},\\[2ex]
\sqrt{\dfrac{2m+1}{2}}2^{k/2}\left[\displaystyle\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}\right. & \\
\left.\times(-1)^{d-g}2^{(k-1)g}\,\Gamma(g+1)\dfrac{\left(t-\dfrac{2n-2}{2^k}\right)^{\alpha+g}}{\Gamma(\alpha+g+1)}\right], & \text{if} \quad \dfrac{2n-2}{2^k}\le t<\dfrac{2n}{2^k},\\[2ex]
\sqrt{\dfrac{2m+1}{2}}2^{k/2}\left[\displaystyle\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}\right. & \\
\times 2^{(k-1)g}\,\Gamma(g+1)\dfrac{\left(t-\dfrac{2n-2}{2^k}\right)^{\alpha+g}}{\Gamma(\alpha+g+1)}-\displaystyle\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d} & \\
\left.\times 2^{(k-1)d}\,\Gamma(d+1)\dfrac{\left(t-\dfrac{2n}{2^k}\right)^{\alpha+d}}{\Gamma(\alpha+d+1)}\right], & \text{if} \quad \dfrac{2n}{2^k}\le t.
\end{cases}
\tag{2.22}
$$

Proof. Using the unit step function, we can write Legendre wavelets as follows:

$$
\psi_{n,m}(t)=\sqrt{\frac{2m+1}{2}}2^{k/2}\left(v_{\frac{2n-2}{2^k}}(t)L_m(2^kt-2n+1)-v_{\frac{2n}{2^k}}(t)L_m(2^kt-2n+1)\right), \tag{2.23}
$$

where $v_a(t)$ is the unit step function given by

$$
v_a(t)=\begin{cases}1, & \text{if} \quad t\ge a,\\ 0, & \text{if} \quad t<a.\end{cases} \tag{2.24}
$$

We make use of the Laplace transform to get $I^{\alpha}\psi_{n,m}(t)$. The Laplace transform has the following property

$$
\mathcal{L}\{v_a(t)f(t)\}=e^{-as}\mathcal{L}\{f(t+a)\}. \tag{2.25}
$$

Using this property, we can write

$$
\begin{aligned}
\mathcal{L}\{\psi_{n,m}(t)\}&=\sqrt{\frac{2m+1}{2}}2^{k/2}\left[e^{-\frac{2n-2}{2^k}s}\mathcal{L}\left\{L_m\left(2^k\left(t+\frac{2n-2}{2^k}\right)-2n+1\right)\right\}\right.\\
&\left.-e^{-\frac{2n}{2^k}s}\mathcal{L}\left\{L_m\left(2^k\left(t+\frac{2n}{2^k}\right)-2n+1\right)\right\}\right]\\
&=\sqrt{\frac{2m+1}{2}}2^{k/2}\left[e^{-\frac{2n-2}{2^k}s}\mathcal{L}\left\{L_m(2^kt-1)\right\}-e^{-\frac{2n}{2^k}s}\mathcal{L}\left\{L_m(2^kt+1)\right\}\right].
\end{aligned} \tag{2.26}
$$

Now, let us find $L_m(2^kt-1)$ and $L_m(2^kt+1)$ by using Equation (2.13)

$$L_m(2^kt-1)=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}\left(\frac{(2^kt-1)-1}{2}\right)^d \tag{2.27}$$

$$=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}\left(2^{k-1}t-1\right)^d \tag{2.28}$$

$$=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}\sum_{g=0}^{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}t^g \tag{2.29}$$

$$=\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}t^g. \tag{2.30}$$

Similarly, we have

$$L_m(2^kt+1)=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}\left(\frac{(2^kt+1)-1}{2}\right)^d \tag{2.31}$$

$$=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}t^d. \tag{2.32}$$

By taking the Laplace transform of Equation (2.30), we get

$$\mathcal{L}\{L_m(2^kt-1)\}=\mathcal{L}\left\{\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}t^g\right\} \tag{2.33}$$

$$=\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\mathcal{L}\{t^g\} \tag{2.34}$$

$$=\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\frac{\Gamma(g+1)}{s^{g+1}}. \tag{2.35}$$

Likewise, we take the Laplace transform of Equation (2.32).

$$\mathcal{L}\{L_m(2^kt+1)\}=\mathcal{L}\left\{\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}t^d\right\} \tag{2.36}$$

$$=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\mathcal{L}\left\{t^d\right\} \tag{2.37}$$

$$=\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\frac{\Gamma(d+1)}{s^{d+1}}. \tag{2.38}$$

Substituting Equations (2.35) and (2.38) into Equation (2.26), we obtain

$$\begin{aligned}\mathcal{L}\left\{\psi_{n,m}(t)\right\}&=\sqrt{\frac{2m+1}{2}}2^{k/2}\left[e^{-\frac{2n-2}{2^k}s}\mathcal{L}\left\{L_m(2^kt-1)\right\}-e^{-\frac{2n}{2^k}s}\mathcal{L}\left\{L_m(2^kt+1)\right\}\right]\\&=\sqrt{\frac{2m+1}{2}}2^{k/2}\left[e^{-\frac{2n-2}{2^k}s}\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\frac{\Gamma(g+1)}{s^{g+1}}\right.\\&\left.-e^{-\frac{2n}{2^k}s}\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\frac{\Gamma(d+1)}{s^{d+1}}\right]\end{aligned}$$

$$= \sqrt{\frac{2m+1}{2}}2^{k/2}\left[\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\frac{\Gamma(g+1)e^{-\frac{2n-2}{2^k}s}}{s^{g+1}} - \sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\frac{\Gamma(d+1)e^{-\frac{2n}{2^k}s}}{s^{d+1}}\right]. \tag{2.39}$$

For a function $f(t)$, we can write its Riemann-Liouville integral of order α in terms of the convolution operator:

$$_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t), \tag{2.40}$$

where $*$ represents the convolution product of $t^{\alpha-1}$ and $f(t)$. Since

$$_0I_t^\alpha \psi_{n,m}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * \psi_{n,m}(t), \tag{2.41}$$

we can write

$$\mathcal{L}\left\{_0I_t^\alpha \psi_{n,m}(t)\right\} = \mathcal{L}\left\{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right\}\mathcal{L}\left\{\psi_{n,m}(t)\right\} = \frac{1}{s^\alpha}\mathcal{L}\left\{\psi_{n,m}(t)\right\}. \tag{2.42}$$

By substituting Equation (2.39) into Equation (2.42), we get

$$\begin{aligned}\mathcal{L}\left\{_0I_t^\alpha \psi_{n,m}(t)\right\} &= \frac{1}{s^\alpha}\sqrt{\frac{2m+1}{2}}2^{k/2}\left[\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\frac{\Gamma(g+1)e^{-\frac{2n-2}{2^k}s}}{s^{g+1}}\right.\\ &\quad\left.-\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\frac{\Gamma(d+1)e^{-\frac{2n}{2^k}s}}{s^{d+1}}\right]\\ &= \frac{1}{s^\alpha}\sqrt{\frac{2m+1}{2}}2^{k/2}\left[\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\Gamma(g+1)\frac{e^{-\frac{2n-2}{2^k}s}}{s^{g+1}}\right.\\ &\quad\left.-\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\Gamma(d+1)\frac{e^{-\frac{2n}{2^k}s}}{s^{d+1}}\right]\\ &= \sqrt{\frac{2m+1}{2}}2^{k/2}\left[\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\,\Gamma(g+1)\frac{e^{-\frac{2n-2}{2^k}s}}{s^{\alpha+g+1}}\right.\\ &\quad\left.-\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\,\Gamma(d+1)\frac{e^{-\frac{2n}{2^k}s}}{s^{\alpha+d+1}}\right].\end{aligned} \tag{2.43}$$

Applying the inverse Laplace transform to both sides yields $_0I_t^\alpha \psi_{n,m}(t)$,

$$\begin{aligned}{}_0I_t^\alpha \psi_{n,m}(t) &= \sqrt{\frac{2m+1}{2}}2^{k/2}\left[\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\,\Gamma(g+1)\mathcal{L}^{-1}\left\{\frac{e^{-\frac{2n-2}{2^k}s}}{s^{\alpha+g+1}}\right\}\right.\\ &\quad\left.-\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\,\Gamma(d+1)\mathcal{L}^{-1}\left\{\frac{e^{-\frac{2n}{2^k}s}}{s^{\alpha+d+1}}\right\}\right]\\ &= \sqrt{\frac{2m+1}{2}}2^{k/2}\left[\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g}2^{(k-1)g}\,\Gamma(g+1)\right.\end{aligned} \tag{2.44}$$

$$\times \frac{v_{\frac{2n-2}{2^k}}(t)\left(t-\frac{2n-2}{2^k}\right)^{\alpha+g}}{\Gamma(\alpha+g+1)} - \sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d}2^{(k-1)d}\times\Gamma(d+1)\frac{v_{\frac{2n}{2^k}}(t)\left(t-\frac{2n}{2^k}\right)^{\alpha+d}}{\Gamma(\alpha+d+1)}\Bigg].$$

We can rewrite the Riemann-Liouville integral of order α of $\psi_{n,m}$ as a piecewise function.

$$ {}_0I_t^{\alpha}\psi_{n,m}(t) = \begin{cases} 0, & \text{if } \ t<\dfrac{2n-2}{2^k}, \\ \sqrt{\dfrac{2m+1}{2}}2^{k/2}\Bigg[\displaystyle\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g} & \\ \quad\times(-1)^{d-g}2^{(k-1)g}\,\Gamma(g+1)\dfrac{\left(t-\frac{2n-2}{2^k}\right)^{\alpha+g}}{\Gamma(\alpha+g+1)}\Bigg], & \text{if } \ \dfrac{2n-2}{2^k}\le t<\dfrac{2n}{2^k}, \\ \sqrt{\dfrac{2m+1}{2}}2^{k/2}\Bigg[\displaystyle\sum_{d=0}^{m}\sum_{g=0}^{d}\binom{m}{d}\binom{m+d}{d}\binom{d}{g}(-1)^{d-g} & \\ \quad\times 2^{(k-1)g}\,\Gamma(g+1)\dfrac{\left(t-\frac{2n-2}{2^k}\right)^{\alpha+g}}{\Gamma(\alpha+g+1)}-\displaystyle\sum_{d=0}^{m}\binom{m}{d}\binom{m+d}{d} & \\ \quad\times 2^{(k-1)d}\,\Gamma(d+1)\dfrac{\left(t-\frac{2n}{2^k}\right)^{\alpha+d}}{\Gamma(\alpha+d+1)}\Bigg], & \text{if } \ \dfrac{2n}{2^k}\le t. \end{cases} \tag{2.45}$$

2.4. **Quasilinearization.** The time-fractional Fisher Equation (1.1) is a nonlinear equation. For ease of computation, we should linearize it. For this purpose, we employ the quasilinearization technique, which was presented by Bellman and Kalaba [4]. First, we need to rewrite the modified time-fractional Fisher Equation (1.1) in the following form

$$L\left(y,Dy,D^2y\right)+N\left(y,Dy,D^2y\right)+q(x,t)=\frac{\partial^{\alpha}y(x,t)}{\partial t^{\alpha}}, \tag{2.46}$$

where Dy and D^2y denote the partial derivatives $\frac{\partial y}{\partial x}$ and $\frac{\partial^2 y}{\partial x^2}$, respectively. $L\left(y,Dy,D^2y\right)$ consists of the linear terms and $N\left(y,Dy,D^2y\right)$ consists of the nonlinear terms of the equation. For the time-fractional Fisher Equation (1.1), the linear part is

$$L\left(y,Dy,D^2y\right)=\delta\frac{\partial^2 y(x,t)}{\partial x^2}+\lambda y(x,t), \tag{2.47}$$

and the nonlinear part is

$$N\left(y,Dy,D^2y\right)=-\lambda y^{p+1}(x,t). \tag{2.48}$$

Suppose we have an initial guess y_0 for the solution of the time-fractional Fisher Equation (2.46). The nonlinear part N can be approximated by expanding the Taylor series around the initial guess y_0 and using only linear terms as follows:

$$
\begin{aligned}
N\left(y, Dy, D^2y\right) &\approx N\left(y_0, Dy_0, D^2y_0\right) + \frac{\partial}{\partial y} N\left(y_0, Dy_0, D^2y_0\right)(y - y_0) \\
&\quad + \frac{\partial}{\partial\, Dy} N\left(y_0, Dy_0, D^2y_0\right)(Dy - Dy_0) + \frac{\partial}{\partial\, D^2y} N\left(y_0, Dy_0, D^2y_0\right)(D^2y - D^2y_0).
\end{aligned} \tag{2.49}
$$

Here, $\frac{\partial}{\partial\, D^2y} N\left(y_0, Dy_0, D^2y_0\right)$ means that we first take the partial derivative of the nonlinear part N with respect to D^2y and then evaluate the result for (y_0, Dy_0, D^2y_0). Similarly, $\frac{\partial}{\partial\, Dy} N\left(y_0, Dy_0, D^2y_0\right)$ means that we first take the partial derivative of the nonlinear part N with respect to Dy and then evaluate the result for (y_0, Dy_0, D^2y_0). It should be noted that the right-hand side of approximation (2.49) is linear. Now, we can replace the nonlinear part of the time-fractional Fisher Equation (2.46) by the right-hand side of the approximation (2.49). Then we can solve the resulting linear equation for y and call the solution y_1. Again, the nonlinear part N can be approximated by expanding the Taylor series around the approximate solution y_1 and using only linear terms. Using this approximation, we can solve the resulting linear equation for y and call it y_2. Continuing this way, the general method for the $(r+1)$-th iteration can be written as follows

$$
\begin{aligned}
N\left(y, Dy, D^2y\right) &\approx N\left(y_r, Dy_r, D^2y_r\right) + \frac{\partial}{\partial y} N\left(y_r, Dy_r, D^2y_r\right)(y_{r+1} - y_r) \\
&\quad + \frac{\partial}{\partial\, Dy} N\left(y_r, Dy_r, D^2y_r\right)(Dy_{r+1} - Dy_r) \\
&\quad + \frac{\partial}{\partial\, D^2y} N\left(y_r, Dy_r, D^2y_r\right)(D^2y_{r+1} - D^2y_r).
\end{aligned} \tag{2.50}
$$

Since $N\left(y, Dy, D^2y\right) = -\lambda y^{p+1}$ for the time-fractional Fisher Equation (2.46), we have

$$
-\lambda y^{p+1} \approx -\lambda y_r^{p+1} - \lambda\left((p+1)(y_r^p)\right)(y_{r+1} - y_r) \tag{2.51}
$$

$$
= -\lambda y_r^{p+1} - \lambda(p+1) y_r^p y_{r+1} + \lambda(p+1) y_r^{p+1}. \tag{2.52}
$$

So, in each iteration, we need to solve the equation

$$
\delta \frac{\partial^2 y_{r+1}(x,t)}{\partial x^2} + \lambda y_{r+1}(x,t) + q(x,t) - \lambda(p+1) y_r^p(x,t) y_{r+1}(x,t) + \lambda p y_r^{p+1}(x,t) = \frac{\partial^\alpha y_{r+1}(x,t)}{\partial t^\alpha}, \tag{2.53}
$$

which can be rearranged as

$$
\delta \frac{\partial^2 y_{r+1}(x,t)}{\partial x^2} + \left(\lambda - \lambda(p+1) y_r^p(x,t)\right) y_{r+1}(x,t) + q(x,t) + \lambda p y_r^{p+1}(x,t) = \frac{\partial^\alpha y_{r+1}(x,t)}{\partial t^\alpha}, \tag{2.54}
$$

with the initial condition

$$
y_{r+1}(x,0) = w(x), \tag{2.55}
$$

and boundary conditions

$$
y_{r+1}(0,t) = z_1(t), \tag{2.56}
$$

$$
y_{r+1}(1,t) = z_2(t). \tag{2.57}
$$

Furthermore, Bellman and Kalaba [4] showed that the sequence $\{y_{r+1}\}_{r=0}^{\infty}$ converges quadratically to y if there is a convergence. As with the Newton-Raphson method for approximating the roots of algebraic equations, the initial guess has a very crucial impact on the convergence of the quasilinearization technique.

3. Convergence and Error Analysis

Theorem 3.1. *[17] Let $y = y(x,t)$ be a square-integrable function on $[0,1) \times [0,1)$, that is $y(x,t) \in L^2([0,1) \times [0,1))$. Suppose that $y(x,t)$ has the property $\left|\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2}\right| \leq K$, where $K \in \mathbb{R}^+$. Then we have the upper bound*

$$|a_{n,m,p,q}| \leq \frac{12K}{n^{5/2}(2m-3)^2\, p^{5/2}(2q-3)^2}, \tag{3.1}$$

for $|a_{n,m,p,q}|$. Moreover, the Legendre wavelet series expansion of $y(x,t)$ converges uniformly to $y(x,t)$.

Proof. Please see [17]. □

Maleknejad *et al.* [17] gave an upper bound for the approximation error when $k = h$ and $M = Q$. Using similar steps, we generalized the upper bound for the approximation error. In our error estimation, k need not be equal to h and M need not be equal to Q.

Theorem 3.2. *Let $y = y(x,t)$ be a square-integrable function on $[0,1) \times [0,1)$, that is $y(x,t) \in L^2([0,1) \times [0,1))$. Suppose that $y(x,t)$ has the property $\left|\frac{\partial^4 y(x,t)}{\partial x^2 \partial t^2}\right| \leq K$, where $K \in \mathbb{R}^+$. Then the approximation error can be bounded by the following error estimation*

$$\|y(x,t) - y_{k,M,h,Q}(x,t)\|_2 \leq \frac{K}{(2^{k-1})^2(M-1)^{3/2}(2^{h-1})^2(Q-1)^{3/2}}. \tag{3.2}$$

Proof.

$$\|y(x,t) - y_{k,M,h,Q}(x,t)\|_2 = \left(\int_0^1\int_0^1\left(y(x,t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{p=1}^{2^{h-1}}\sum_{q=0}^{Q-1} a_{n,m,p,q} \times \psi_{n,m}(x)\psi_{p,q}(t)\right)^2 dx\,dt\right)^{1/2}.$$

Note that

$$\begin{aligned}
y(x,t) &- \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{p=1}^{2^{h-1}}\sum_{q=0}^{Q-1} a_{n,m,p,q}\psi_{n,m}(x)\psi_{p,q}(t) \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{p=1}^{\infty}\sum_{q=0}^{\infty} a_{n,m,p,q}\psi_{n,m}(x)\psi_{p,q}(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{p=1}^{2^{h-1}}\sum_{q=0}^{Q-1} a_{n,m,p,q}\psi_{n,m}(x)\psi_{p,q}(t) \\
&= \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\sum_{p=2^{h-1}+1}^{\infty}\sum_{q=Q}^{\infty} a_{n,m,p,q}\psi_{n,m}(x)\psi_{p,q}(t).
\end{aligned} \tag{3.3}$$

Thus, we can write

$$\|y(x,t) - y_{k,M,h,Q}(x,t)\|_2 = \left(\int_0^1\int_0^1\left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\sum_{p=2^{h-1}+1}^{\infty}\sum_{q=Q}^{\infty} a_{n,m,p,q} \times \psi_{n,m}(x)\psi_{p,q}(t)\right)^2 dx\,dt\right)^{1/2}.$$

Due to orthonormality (2.15), we have

$$\begin{aligned}
\|y(x,t) - y_{k,M,h,Q}(x,t)\|_2 &= \left(\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=M}^{\infty}\sum_{p=2^{h-1}+1}^{\infty}\sum_{q=Q}^{\infty} a^2_{n,m,p,q} \times \int_0^1\int_0^1 \psi^2_{n,m}(x)\psi^2_{p,q}(t)\,dx\,dt\right)^{1/2} \\
&\leq 12K\left(\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^5}\sum_{m=M}^{\infty}\frac{1}{(2m-3)^4}\sum_{p=2^{h-1}+1}^{\infty}\frac{1}{p^5}\sum_{q=Q}^{\infty}\frac{1}{(2q-3)^4}\right)^{1/2}.
\end{aligned} \tag{3.4}$$

It is known that

$$\sum_{u=v}^{\infty} \frac{1}{u^w} \le \frac{1}{(w-1)(v-1)^{(w-1)}}. \tag{3.5}$$

Thus, we have

$$\sum_{n=2^{k-1}+1}^{\infty} \frac{1}{n^5} \le \frac{1}{4(2^{k-1})^4}, \tag{3.6}$$

$$\sum_{p=2^{h-1}+1}^{\infty} \frac{1}{p^5} \le \frac{1}{4(2^{h-1})^4}. \tag{3.7}$$

Note that

$$\sum_{m=M}^{\infty} \frac{1}{(2m-3)^4} \le \sum_{m=M}^{\infty} \frac{1}{m^4}, \qquad \text{for } m \ge 3, \tag{3.8}$$

$$\sum_{q=Q}^{\infty} \frac{1}{(2q-3)^4} \le \sum_{q=Q}^{\infty} \frac{1}{q^4}, \qquad \text{for } q \ge 3. \tag{3.9}$$

Therefore, we can write

$$\sum_{m=M}^{\infty} \frac{1}{(2m-3)^4} \le \frac{1}{3(M-1)^3}, \tag{3.10}$$

$$\sum_{q=Q}^{\infty} \frac{1}{(2q-3)^4} \le \frac{1}{3(Q-1)^3}. \tag{3.11}$$

As a result, we conclude that

$$\begin{aligned} \|y(x,t) - y_{k,M,h,Q}(x,t)\|_2 &\le 12K \left(\sum_{n=2^{k-1}+1}^{\infty} \frac{1}{n^5} \sum_{m=M}^{\infty} \frac{1}{(2m-3)^4} \sum_{p=2^{h-1}+1}^{\infty} \frac{1}{p^5} \sum_{q=Q}^{\infty} \frac{1}{(2q-3)^4} \right)^{1/2} \\ &\le \frac{K}{(2^{k-1})^2 (M-1)^{3/2} (2^{h-1})^2 (Q-1)^{3/2}}. \end{aligned} \tag{3.12}$$

□

4. Description of the Proposed Method

Let $y_{r+1}(x,t)$ be the approximate solution of the nonlinear time-fractional Fisher Equation (1.1) obtained by the quasilinearization technique at the $(r+1)$-th iteration. Using Legendre wavelets, we can approximate $\frac{\partial^{2+\alpha} y_{r+1}(x,t)}{\partial x^2 \partial t^\alpha}$ by

$$\frac{\partial^{2+\alpha} y_{r+1}(x,t)}{\partial x^2 \partial t^\alpha} \approx \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \psi_i(x) \psi_j(t). \tag{4.1}$$

Taking the α-th order fractional integral of both sides of Equation (4.1) with respect to t, we obtain

$$\frac{\partial^2 y_{r+1}(x,t)}{\partial x^2} \approx \left. \frac{\partial^2 y_{r+1}(x,t)}{\partial x^2} \right|_{t=0} + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \psi_i(x) \left[{}_0I_t^\alpha \psi_j(t) \right]. \tag{4.2}$$

Since $\left.\dfrac{\partial^2 y_{r+1}(x,t)}{\partial x^2}\right|_{t=0} = y''_{r+1}(x,0) = w''(x)$ by the initial condition, we can write

$$\frac{\partial^2 y_{r+1}(x,t)}{\partial x^2} \approx w''(x) + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \psi_i(x) \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \tag{4.3}$$

Integrating Equation (4.3) with respect to x from 0 to x, we get

$$\frac{\partial y_{r+1}(x,t)}{\partial x} \approx \left.\frac{\partial y_{r+1}(x,t)}{\partial x}\right|_{x=0} + w'(x) - w'(0) + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{1} \psi_i(x)\right] \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \tag{4.4}$$

Again, integrating Equation (4.4) with respect to x from 0 to x, we get

$$\begin{aligned} y_{r+1}(x,t) \approx\; & y_{r+1}(0,t) + x\left(\left.\frac{\partial y_{r+1}(x,t)}{\partial x}\right|_{x=0}\right) + w(x) - w(0) - xw'(0) \\ & + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right] \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \end{aligned} \tag{4.5}$$

Now, let us evaluate Equation (4.5) at $x = 1$:

$$\begin{aligned} y_{r+1}(1,t) \approx\; & y_{r+1}(0,t) + 1\left(\left.\frac{\partial y_{r+1}(x,t)}{\partial x}\right|_{x=0}\right) + w(1) - w(0) - 1w'(0) \\ & + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right]_{x=1} \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \end{aligned} \tag{4.6}$$

Using the boundary conditions, we can write

$$z_2(t) \approx z_1(t) + \left(\left.\frac{\partial y_{r+1}(x,t)}{\partial x}\right|_{x=0}\right) + w(1) - w(0) - w'(0) + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right]_{x=1} \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \tag{4.7}$$

Thus, we have

$$\left(\left.\frac{\partial y_{r+1}(x,t)}{\partial x}\right|_{x=0}\right) \approx z_2(t) - z_1(t) + w(0) + w'(0) - w(1) - \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right]_{x=1} \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \tag{4.8}$$

If we substitute Equation (4.8) into Equation (4.5), we get

$$\begin{aligned} y_{r+1}(x,t) \approx\; & z_1(t) + x\left(z_2(t) - z_1(t) + w(0) + w'(0) - w(1) - \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j} \left[{}_0I_x^{2} \psi_i(x)\right]_{x=1} \left[{}_0I_t^{\alpha} \psi_j(t)\right]\right) \\ & + w(x) - w(0) - xw'(0) + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right] \left[{}_0I_t^{\alpha} \psi_j(t)\right]. \end{aligned} \tag{4.9}$$

Taking the α-th order Caputo derivative of both sides of Equation (4.9) with respect to t yields

$$\begin{aligned} \frac{\partial^{\alpha} y_{r+1}(x,t)}{\partial t^{\alpha}} \approx\; & \left[D_t^{\alpha} z_1(t)\right] + x\left(\left[D_t^{\alpha} z_2(t)\right] - \left[D_t^{\alpha} z_1(t)\right] - \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right]_{x=1} \psi_j(t)\right) \\ & + \sum_{i=1}^{2^{k-1}M} \sum_{j=1}^{2^{h-1}Q} a_{i,j}^{r+1} \left[{}_0I_x^{2} \psi_i(x)\right] \psi_j(t). \end{aligned} \tag{4.10}$$

If we substitute Equations (4.3), (4.9), and (4.10) into Equation (2.53), and replace $\approx$ by $=$, we get

$$\begin{aligned}
&\delta\left[w''(x)+\sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{h-1}Q}a_{i,j}^{r+1}\psi_i(x)\left[{}_0I_t^{\alpha}\psi_j(t)\right]\right]+\left(\lambda-\lambda(p+1)y_r^p(x,t)\right)\\
&\times\left[z_1(t)+x\left(z_2(t)-z_1(t)+w(0)+w'(0)-w(1)-\sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{h-1}Q}a_{i,j}^{r+1}\left[{}_0I_x^{2}\psi_i(x)\right]_{x=1}\left[{}_0I_t^{\alpha}\psi_j(t)\right]\right)\right.\\
&\left.+w(x)-w(0)-xw'(0)+\sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{h-1}Q}a_{i,j}^{r+1}\left[{}_0I_x^{2}\psi_i(x)\right]\left[{}_0I_t^{\alpha}\psi_j(t)\right]\right]+q(x,t)+\lambda p y_r^{p+1}(x,t) \qquad (4.11)\\
&=\left[D_t^{\alpha}z_1(t)\right]+x\left(\left[D_t^{\alpha}z_2(t)\right]-\left[D_t^{\alpha}z_1(t)\right]-\sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{h-1}Q}a_{i,j}^{r+1}\left[{}_0I_x^{2}\psi_i(x)\right]_{x=1}\psi_j(t)\right)\\
&+\sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{h-1}Q}a_{i,j}^{r+1}\left[{}_0I_x^{2}\psi_i(x)\right]\psi_j(t).
\end{aligned}$$

If we solve linear Equation (4.11) at the collocation points $x_e=\dfrac{2e-1}{2^kM}$, $t_f=\dfrac{2f-1}{2^hQ}$, where $e=1,\ldots,2^{k-1}M$, $f=1,\ldots,2^{h-1}Q$, we can obtain the unknown coefficients $a_{i,j}^{r+1}$. We can find an approximate solution by substituting the coefficients $a_{i,j}^{r+1}$ into Equation (4.9).

5. Numerical Examples

We will solve some test problems to investigate the effectiveness of the Legendre wavelet collocation method with the quasilinearization technique (LWCMQT). In addition, the maximum absolute errors

$$E_{L_\infty}:=\max\left|y_{exact}(x_e,t_f)-y_{approximate}(x_e,t_f)\right|, \qquad (5.1)$$

where the maximum is taken over all collocation points (x_e,t_f), will be compared with the Haar wavelet collocation iteration method (HWCIM) [3] and the modified variational iteration method (MVIM) [20]. In all examples, we will take the resolution levels $k=2$, $h=2$, and $M=2$, $Q=2$, and iterate the quasilinearization technique three times. All calculations and graphs were obtained using Wolfram Mathematica Online [13].

Example 5.1. *In the first test problem, we consider the following time-fractional homogeneous Fisher equation*

$$\frac{\partial^{\alpha}y(x,t)}{\partial t^{\alpha}}=\frac{\partial^2y(x,t)}{\partial x^2}+y(x,t)\left(1-y^6(x,t)\right), \qquad (5.2)$$

with the initial condition

$$y(x,0)=\left(1+e^{\frac{3x}{2}}\right)^{-\frac{1}{3}}, \qquad (5.3)$$

and boundary conditions

$$y(0,t)=\left(1+e^{\frac{-15t}{4}}\right)^{-\frac{1}{3}}, \qquad (5.4)$$

$$y(1,t)=\left(1+e^{\frac{6-15t}{4}}\right)^{-\frac{1}{3}}. \qquad (5.5)$$

The exact solution for $\alpha=1$ is

$$y_{exact}(x,t)=\left(1+e^{\frac{6x-15t}{4}}\right)^{-\frac{1}{3}}. \qquad (5.6)$$

Table 1. Numerical results for Example 5.1 obtained by using the LWCMQT with resolution levels $k = 2$, $h = 2$, and $M = 2$, $Q = 2$.

Col. Pts. (x,t)	$\alpha = 0.5$ $y_3(x,t)$	$\alpha = 0.7$ $y_3(x,t)$	$\alpha = 0.9$ $y_3(x,t)$	$\alpha = 1$ $y_3(x,t)$	*Absolute error for* $\alpha = 1$ $\lvert y_{exact}(x,t) - y_3(x,t)\rvert$
$(\frac{1}{8}, \frac{1}{8})$	0.841965	0.837217	0.831816	0.829149	$8.64386E-05$
$(\frac{1}{8}, \frac{3}{8})$	0.920330	0.919341	0.918173	0.917268	$2.83358E-05$
$(\frac{1}{8}, \frac{5}{8})$	0.960751	0.961107	0.962609	0.964099	$4.83664E-05$
$(\frac{1}{8}, \frac{7}{8})$	0.980168	0.981730	0.983990	0.985399	$6.85387E-05$
$(\frac{3}{8}, \frac{1}{8})$	0.809699	0.799151	0.787212	0.781343	$2.34064E-04$
$(\frac{3}{8}, \frac{3}{8})$	0.895346	0.893036	0.889976	0.887503	$8.54197E-05$
$(\frac{3}{8}, \frac{5}{8})$	0.942496	0.943067	0.946154	0.949353	$8.42780E-05$
$(\frac{3}{8}, \frac{7}{8})$	0.967799	0.971121	0.975982	0.979079	$1.44718E-04$
$(\frac{5}{8}, \frac{1}{8})$	0.757606	0.746537	0.733928	0.727673	$2.50099E-04$
$(\frac{5}{8}, \frac{3}{8})$	0.859489	0.856715	0.853110	0.850308	$1.36321E-04$
$(\frac{5}{8}, \frac{5}{8})$	0.922621	0.923055	0.926167	0.929488	$5.55606E-05$
$(\frac{5}{8}, \frac{7}{8})$	0.957987	0.961485	0.966695	0.970038	$1.22053E-04$
$(\frac{7}{8}, \frac{1}{8})$	0.685000	0.679575	0.673283	0.670091	$1.03955E-04$
$(\frac{7}{8}, \frac{3}{8})$	0.810705	0.809069	0.807159	0.805823	$8.18697E-05$
$(\frac{7}{8}, \frac{5}{8})$	0.900009	0.900163	0.901686	0.903333	$1.13899E-05$
$(\frac{7}{8}, \frac{7}{8})$	0.951098	0.952903	0.955655	0.957412	$4.45979E-05$

Table 2. Maximum absolute errors (E_{L_∞}) of the LWCMQT, HWCIM [3], and MVIM [20] for the numerical solution of Example 5.1.

The Method	*Maximum Absolute Error* (E_{L_∞})
LWCMQT	$2.50099E-04$
LWAM [1]	$2.2554E-04$
HWCIM [3]	$1.17E-03$
MVIM [20]	$1.97465E-01$

We use $y_0(x,t) = \left(1 + e^{\frac{3x}{2}}\right)^{-\frac{1}{3}}$ *as an initial guess, and implement the Legendre wavelet collocation method with the quasilinearization technique. We iterate the quasilinearization technique three times. The numerical results for resolution levels* $k = 2$, $h = 2$, *and* $M = 2$, $Q = 2$ *are given in Table 1. In Table 2, the maximum absolute errors of some other methods are compared. The approximate solution is plotted in Figure 1. Also, the absolute error is depicted in Figure 2.*

Example 5.2. *In the second test problem, we consider the following time-fractional homogeneous Fisher equation*

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \frac{\partial^2 y(x,t)}{\partial x^2} + y(x,t)\left(1 - y(x,t)\right), \tag{5.7}$$

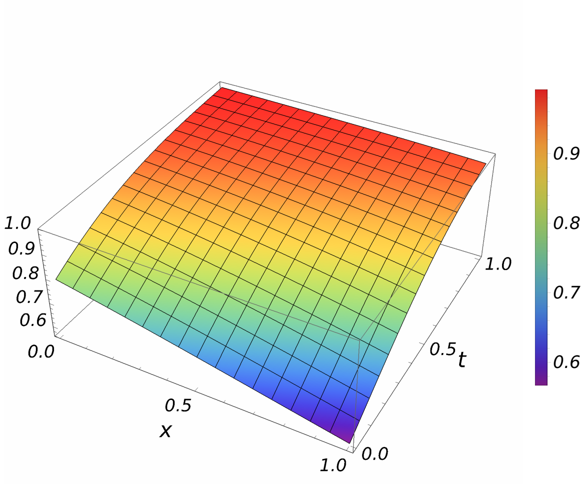

FIGURE 1. Approximate solution of Example 5.1 for $\alpha = 1$ obtained by the LWCMQT with resolution levels $k = 2$, $h = 2$ and $M = 2$, $Q = 2$ at the 3^{rd} iteration.

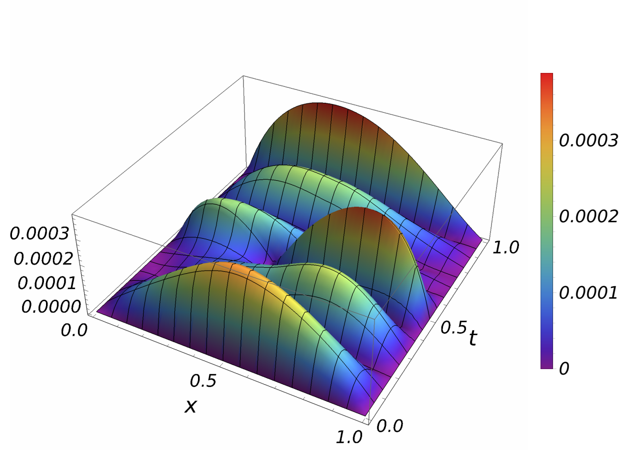

FIGURE 2. The absolute error $|y_{exact}(x,t) - y_3(x,t)|$ of Example 5.1.

with the initial condition

$$y(x,0) = \beta, \tag{5.8}$$

and boundary conditions

$$y(0,t) = \frac{\beta e^t}{1-\beta+\beta e^t}, \tag{5.9}$$

$$y(1,t) = \frac{\beta e^t}{1-\beta+\beta e^t}, \tag{5.10}$$

where β is a constant. The exact solution for $\alpha = 1$ is given by

$$y_{exact}(x,t) = \frac{\beta e^t}{1-\beta+\beta e^t}. \tag{5.11}$$

We use $y_0(x,t) = \beta$ as an initial guess and implement the Legendre wavelet collocation method with the quasilinearization technique. We iterate the quasilinearization technique three times. The numerical results for $\beta = \frac{2}{3}$, $k = 2$, $h = 2$, and $M = 2$, $Q = 2$ are given in Table 3. In Table 4, the maximum absolute errors of some other methods are compared. The approximate solution is plotted in Figure 3. Also, the absolute error is depicted in Figure 4.

Example 5.3. *In the third test problem, we consider the following time-fractional non-homogeneous Fisher equation*

$$\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \frac{\partial^2 y(x,t)}{\partial x^2} + y(x,t)\left(1 - y^3(x,t)\right) + q(x,t), \tag{5.12}$$

where

$$q(x,t) = t\left(-2 - x(t+x)\left(1 - t^3x^3(t+x)^3\right) + \frac{x^2 t^{-\alpha}}{\Gamma(2-\alpha)} + \frac{2xt^{1-\alpha}}{\Gamma(3-\alpha)}\right), \tag{5.13}$$

with the initial condition

$$y(x,0) = 0, \tag{5.14}$$

TABLE 3. Numerical results for Example 5.2 obtained by using the LWCMQT with resolution levels $k = 2$, $h = 2$, and $M = 2$, $Q = 2$.

Col. Pts. (x,t)	$\alpha = 0.5$ $y_3(x,t)$	$\alpha = 0.7$ $y_3(x,t)$	$\alpha = 0.9$ $y_3(x,t)$	$\alpha = 1$ $y_3(x,t)$	*Absolute error for* $\alpha = 1$ $\lvert y_{exact}(x,t) - y_3(x,t)\rvert$
$(\frac{1}{8},\frac{1}{8})$	0.699479	0.697388	0.695016	0.693843	$0.00000E-00$
$(\frac{1}{8},\frac{3}{8})$	0.747179	0.746198	0.745017	0.744244	$0.00000E-00$
$(\frac{1}{8},\frac{5}{8})$	0.789141	0.788652	0.788610	0.788873	$1.11022E-16$
$(\frac{1}{8},\frac{7}{8})$	0.826046	0.826320	0.826995	0.827519	$2.88658E-15$
$(\frac{3}{8},\frac{1}{8})$	0.705598	0.701172	0.696237	0.693843	$0.00000E-00$
$(\frac{3}{8},\frac{3}{8})$	0.750559	0.748549	0.745995	0.744244	$0.00000E-00$
$(\frac{3}{8},\frac{5}{8})$	0.789508	0.788461	0.788339	0.788873	$4.44089E-16$
$(\frac{3}{8},\frac{7}{8})$	0.824498	0.825061	0.826431	0.827519	$7.77156E-15$
$(\frac{5}{8},\frac{1}{8})$	0.705598	0.701172	0.696237	0.693843	$0.00000E-00$
$(\frac{5}{8},\frac{3}{8})$	0.750559	0.748549	0.745995	0.744244	$0.00000E-00$
$(\frac{5}{8},\frac{5}{8})$	0.789508	0.788461	0.788339	0.788873	$2.22045E-16$
$(\frac{5}{8},\frac{7}{8})$	0.824498	0.825061	0.826431	0.827519	$7.99361E-15$
$(\frac{7}{8},\frac{1}{8})$	0.699479	0.697388	0.695016	0.693843	$0.00000E-00$
$(\frac{7}{8},\frac{3}{8})$	0.747179	0.746198	0.745017	0.744244	$0.00000E-00$
$(\frac{7}{8},\frac{5}{8})$	0.789141	0.788652	0.788610	0.788873	$2.22045E-16$
$(\frac{7}{8},\frac{7}{8})$	0.826046	0.826320	0.826995	0.827519	$2.88658E-15$

TABLE 4. Maximum absolute errors (E_{L_∞}) of the LWCMQT, HWCIM [3], and MVIM [20] for the numerical solution of Example 5.2.

The Method	*Maximum Absolute Error* (E_{L_∞})
LWCMQT	$7.99361E-15$
HWCIM [3]	$2.19E-05$
MVIM [20]	$3.68E-02$
ADM [34]	$3.64588E-03$

and boundary conditions

$$y(0,t) = 0, \tag{5.15}$$

$$y(1,t) = t^2 + t. \tag{5.16}$$

The problem has the exact solution

$$y_{exact}(x,t) = xt^2 + tx^2. \tag{5.17}$$

We use $y_0(x,t) = 0$ *as an initial guess and implement the Legendre wavelet collocation method with the quasilinearization technique. We iterate the quasilinearization technique three times. The numerical results for resolution levels*

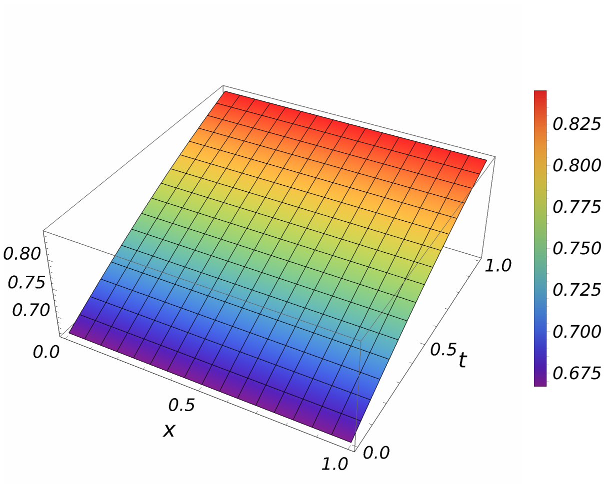

FIGURE 3. Approximate solution of Example 5.2 for $\beta = \dfrac{2}{3}$ and $\alpha = 1$ obtained by the LWCMQT with resolution levels $k = 2$, $h = 2$ and $M = 2$, $Q = 2$ at the 3^{rd} iteration.

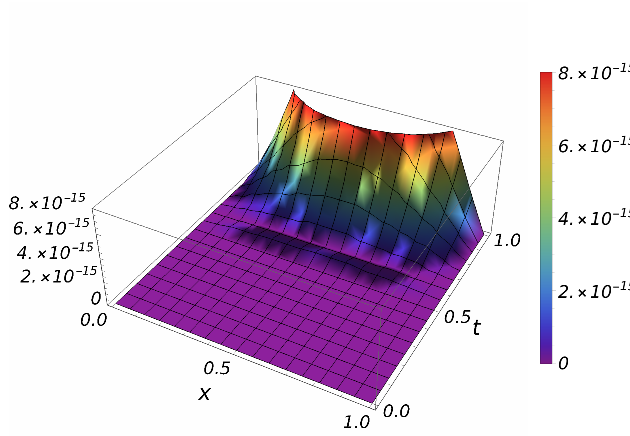

FIGURE 4. The absolute error $|y_{exact}(x,t) - y_3(x,t)|$ of Example 5.2.

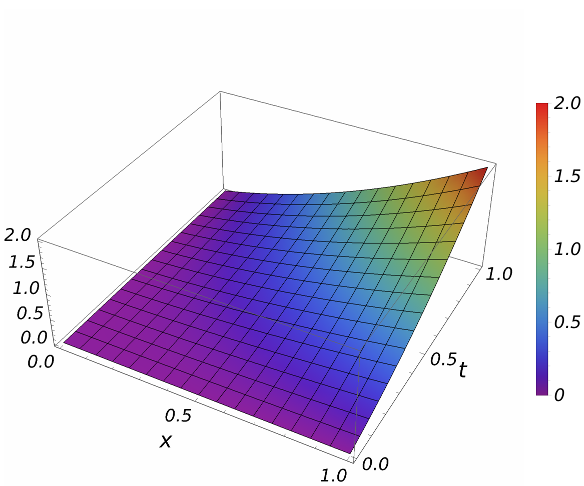

FIGURE 5. Approximate solution of Example 5.3 for $\beta = \dfrac{2}{3}$ and $\alpha = 1$ obtained by the LWCMQT with resolution levels $k = 2$, $h = 2$ and $M = 2$, $Q = 2$ at the 3^{rd} iteration.

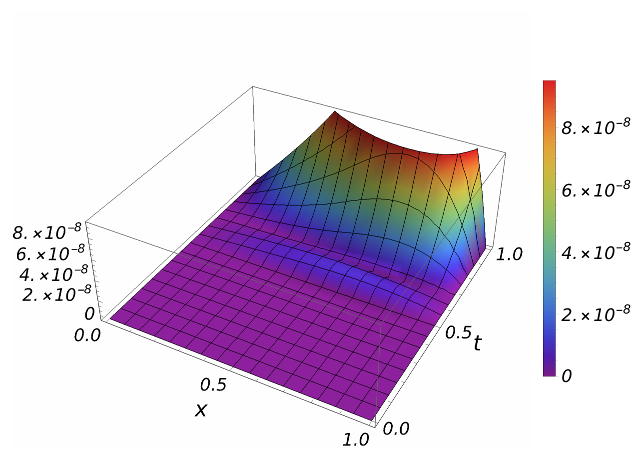

FIGURE 6. The absolute error $|y_{exact}(x,t) - y_3(x,t)|$ of Example 5.3.

$k = 2$, $h = 2$, and $M = 2$, $Q = 2$ are given in Table 5. In Table 6, the maximum absolute errors of some other methods are compared. The approximate solution is plotted in Figure 5. Also, the absolute error is depicted in Figure 6.

TABLE 5. Numerical results for Example 5.3 obtained by using the Legendre wavelet collocation method with the quasilinearization technique with resolution levels $k = 2$, $h = 2$, and $M = 2$, $Q = 2$.

Col. Pts. (x,t)	$\alpha = 0.5$ $y_3(x,t)$	$\alpha = 0.7$ $y_3(x,t)$	$\alpha = 0.9$ $y_3(x,t)$	$\alpha = 1$ $y_3(x,t)$	*Absolute error for* $\alpha = 1$ $\lvert y_{exact}(x,t) - y_3(x,t)\rvert$
$(\frac{1}{8}, \frac{1}{8})$	0.003635	0.003507	0.003635	0.003906	$8.67362E-19$
$(\frac{1}{8}, \frac{3}{8})$	0.023602	0.023678	0.023568	0.023437	$6.93889E-18$
$(\frac{1}{8}, \frac{5}{8})$	0.058457	0.058386	0.058470	0.058593	$2.29585E-09$
$(\frac{1}{8}, \frac{7}{8})$	0.109399	0.109417	0.109399	0.109375	$1.30438E-08$
$(\frac{3}{8}, \frac{1}{8})$	0.022788	0.022480	0.022788	0.023437	$6.93889E-18$
$(\frac{3}{8}, \frac{3}{8})$	0.105861	0.106040	0.105776	0.105469	$0.00000E-00$
$(\frac{3}{8}, \frac{5}{8})$	0.234052	0.233886	0.234089	0.234375	$6.16237E-09$
$(\frac{3}{8}, \frac{7}{8})$	0.410211	0.410251	0.410205	0.410156	$4.49039E-08$
$(\frac{5}{8}, \frac{1}{8})$	0.057944	0.057637	0.057944	0.058593	$6.93889E-18$
$(\frac{5}{8}, \frac{3}{8})$	0.234767	0.234946	0.234681	0.234375	$5.55112E-17$
$(\frac{5}{8}, \frac{5}{8})$	0.487962	0.487799	0.487999	0.488281	$7.27352E-09$
$(\frac{5}{8}, \frac{7}{8})$	0.820365	0.820403	0.820359	0.820313	$8.52705E-08$
$(\frac{7}{8}, \frac{1}{8})$	0.109105	0.108976	0.109104	0.109375	$1.38779E-17$
$(\frac{7}{8}, \frac{3}{8})$	0.410320	0.410396	0.410287	0.410156	$1.11022E-16$
$(\frac{7}{8}, \frac{5}{8})$	0.820182	0.820114	0.820194	0.820312	$3.49513E-09$
$(\frac{7}{8}, \frac{7}{8})$	1.339860	1.339880	1.339870	1.339840	$6.49032E-08$

TABLE 6. Maximum absolute errors (E_{L_∞}) of the LWCMQT, and HWCIM [3] for the numerical solution of Example 5.3.

The Method	*Maximum Absolute Error* (E_{L_∞})
LWCMQT	$8.52705E-08$
HWCIM [3]	$1.19E-03$

6. Conclusion

In this paper, we proposed a numerical method based on Legendre wavelets and the quasilinearization technique for the time-fractional Fisher equation. We gave a formula for the Riemann-Liouville integral of Legendre wavelets. To investigate the efficiency of the proposed method, three numerical examples were solved. The examples show that the proposed method is quite effective even when the resolution levels k, h and the degree of polynomials M, Q are very small.

Declarations

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- **Conflict of interest/Competing interests**: The authors have no competing interests to declare that are relevant to the content of this article.

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