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A novel approach using a new generalization of Bernoulli wavelets for solving fractional integro-differential equations with singular kernel

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Abstract

In recent years, numerous fractional-order basis functions have been developed and applied for solving different classes of fractional problems. In this work, a new generalization of fractional-order Bernoulli wavelets is introduced. These new basis functions are used to provide a numerical solution for Hammerstein-type fractional integro-differential equations with a weakly singular kernel. To achieve this, the Riemann-Liouville integral operator is applied to the basis functions, and the result is computed exactly using the analytic form of Bernoulli polynomials. Through this process, key properties of the Riemann-Liouville integral and Caputo derivative are utilized to define two remainders associated with the main problem. After that, using an appropriate set of collocation points, the problem is converted to a system of algebraic equations. Due to the efficiency and high accuracy of this new technique, we extend the method for solving fractional Fredholm-Volterra integro-differential equations. Then, an upper bound of the error is discussed for the approximation of a function based on the fractional-order Bernoulli wavelets. Finally, the method is utilized for solving some illustrative examples to check its performance.

Keywords. Weakly singular fractional integro-differential equations, Generalized fractional-order Bernoulli wavelets, Caputo derivative, Riemann-Liouville integral.

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1. Introduction

Over the past three decades, it has been shown that mathematical modelling of phenomena in science and engineering requires fractional operators [1, 6, 13]. Among these models, fractional integro-differential equations with weakly singular kernel play a crucial role in the modelling of several physical problems. These equations appear in areas such as heat conduction [20], elasticity and fracture mechanics [26], radiative equilibrium [7], etc. Given the importance of this class of equations, the existence and uniqueness of solutions to such equations have been extensively studied (see, e.g., [9, 10]). Since introducing exact solutions to fractional integro-differential equations is impossible in most of the cases, different numerical methods have been developed and used for solving them. In this research, we focus on the following fractional nonlinear Hammerstein type integro-differential equation with weakly singular kernel

$$ {}_0^C D_t^\alpha y(t) = f(t, y(t)) + \int_0^t (t-s)^{-\beta} g(s, y(s)) ds, \qquad t \in [0, T], \tag{1.1}$$

subject to the initial conditions:

$$ y^{(i)}(0) = y_i, \qquad i = 0, 1, \dots, \lceil \alpha \rceil - 1, \tag{1.2}$$

where f and g are linear or nonlinear functions of their arguments, $\alpha > 0$, $0 \le \beta < 1$, ${}_0^C D_t^\alpha$ is the Caputo fractional derivative operator of order α, y_i $(i = 0, \dots, \lceil \alpha \rceil - 1)$ are known real constants, $\lceil \cdot \rceil$ is the ceiling function and y is the unknown function to be determined. We now briefly review some of the existing numerical methods for solving this problem. In [24], piecewise polynomial collocation methods were employed and analyzed for solving problem (1.1)-(1.2)

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with $f(t, y(t)) = q(t) + p(t)y(t)$, where q and p are bounded and continuous functions, and $g(s, y) = y$. An operational matrix technique utilizing the second kind Chebyshev polynomials was introduced for solving the same linear problem in [12]. A linear class of fractional Volterra-Fredholm integro-differential equations with weakly singular kernels has been solved using the CAS wavelet method in [22]. Also, Legendre and second kind Chebyshev wavelet techniques have been presented to solve the same problem in [23] and [21], respectively. In [11], the quadratic hat functions and their operational matrices have been employed for solving problem (1.1)-(1.2) in the case that $f(t, y(t)) = q(t) + p(t)y(t)$ and $g(s, y) = y^m$, where m is a positive integer. The same problem has been solved in [19] using properties of the first kind Chebyshev polynomials. Moreover, a Tayor wavelet-based method has been used to solve a special class of problem (1.1)-(1.2) in [3].

Fractional-order (FO) basis functions have been recently used as an effective tool by many researchers for solving some classes of fractional models. For example, Sabermahani et al. [18] have employed FO Fibonacci-hybrid functions to obtain an approximate solution for fractional delay differential equations. In [25], a numerical scheme based on the FO hybrid Chebyshev functions has been proposed for fractional pantograph differential equations. Postavaru and Toma [15] have solved fractional optimal control problems using the FO hybrid Bernoulli functions. In [14], a generalization of FO hybrid Bernoulli functions has been considered for solving fractional delay differential equations. Also, multi-order fractional differential equations in a general form have been solved by FO hybrid Jacobi functions in [2].

The primary objective of this research is to introduce a new collocation method based on a generalization of FO Bernoulli wavelets (GFOBWs) defined on an arbitrary interval $[0, T)$, for solving problem (1.1)-(1.2). Unlike the work done in [16], where the operational matrix of fractional integration of the FOBWs, defined on $[0, 1)$, has been introduced and used for solving fractional differential equations, we compute here the Riemann-Liouville fractional integral of our basis functions without any error. This result, along with an appropriate set of collocation points, converts the problem under study into a system of algebraic equations. We also extend the method for solving the following fractional integro-differential equations

$$ {}_0^C D_t^{\alpha} y(t) = f\left(t, y(t)\right) + \int_0^t (t-s)^{-\beta} g\left(s, y(s)\right) ds + \int_0^T \kappa\left(t, s, y(s)\right) ds, \qquad t \in [0, T], \tag{1.3} $$

with the same initial conditions given by (1.2) and κ as a given linear or nonlinear continuous function on its arguments defined on $([0, T] \times [0, T] \times \mathbb{R})$. Main advantages of the present work are as follows:

(a) Unlike the operational matrix technique which introduces errors in the computation of the Riemann-Liouville integral of basis functions, the Riemann-Liouville fractional integral of the GFOBWs basis functions is given by the analytic form of Bernoulli polynomials, exactly.
(b) We can improve the accuracy of the numerical solutions by looking for the most suitable fractional order of the basis functions.
(c) In the case of equations whose solutions are non-smooth functions, the accuracy of the numerical approximation of the solution can improve by increasing the level of resolution.

The paper is organized as follows: In section 2, some basic definitions on fractional calculus are presented. The GFOBWs and their main properties are introduced in section 3. Section 4 is devoted to introducing our new method for solving problem (1.1)-(1.2) which includes the computation of Riemann-Liouville fractional integral operator of the GFOBWs basis functions. An extension of the method for solving (1.3) with initial conditions (1.2) is also proposed in this section. In section 5, an error discussion is given. Then, the numerical results are reported in section 6. Finally, we conclude the paper in section 7.

2. Preliminaries on fractional calculus

In this section, we recall two important definitions of fractional calculus which have been widely used due to their applications in recent researches. These definitions include Riemann-Liouville fractional integral and Caputo fractional derivative.

Definition 2.1. Let y be a real valued continuous function defined on $[0,\infty)$ and $\alpha \in \mathbb{R}$ $(n-1<\alpha\leq n,\, n\in\mathbb{N})$. The Caputo fractional derivative of order α of y is given by [8]:

$$ {}_{0}^{C}D_t^{\alpha}y(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}y^{(n)}(s)ds, & n-1<\alpha<n, \\ y^{(n)}(t), & \alpha = n, \end{cases} $$

where $\Gamma(t)$ is the gamma function as

$$ \Gamma(t) = \int_0^{\infty} s^{t-1}e^{-s}ds. $$

Definition 2.2. The left Riemann-Liouville fractional integral ${}_{0}I_t^{\alpha}$ of a function y is defined by [8]:

$$ {}_{0}I_t^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds, \quad t>0, $$

where α is a real positive number.

The following properties of the two above-mentioned operators have important rule in our numerical scheme.

$$ {}_{0}I_t^{\alpha}t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}t^{\nu+\alpha}, \qquad \alpha>0, \quad \nu>-1, \quad t>0, \tag{2.1} $$

$$ {}_{0}I_t^{\alpha}({}_{0}^{C}D_t^{\alpha}y(t)) = y(t) - \sum_{i=0}^{\lceil\alpha\rceil-1} y^{(i)}(0)\frac{t^i}{i!}. \tag{2.2} $$

3. Properties of the GFOBWs

This section is devoted to introducing properties of the GFOBWs defined on an arbitrary interval $[0,T)$.

3.1. **Definition and function approximation.** Before going to the main discussion, we review Bernoulli polynomials.

Definition 3.1. The classical Bernoulli polynomial of order m is defined on the interval $[0,1]$ as

$$ \mathfrak{b}_m(t) = \sum_{r=0}^{m}\binom{m}{r}b_{m-r}t^r, \tag{3.1} $$

where $b_r := \mathfrak{b}_r(0)$, $r=1,\dots,m$, are Bernoulli numbers [17].

Bernoulli polynomials constitute a complete basis for the space $L^2[0,1]$. These polynomials satisfy the following property [17]

$$ \int_0^1 \mathfrak{b}_r(t)\mathfrak{b}_s(t)dt = (-1)^{s-1}\frac{(r!)(s!)}{(r+s)!}b_{r+s}, \quad r,s\geq 1. \tag{3.2} $$

Definition 3.2. The GFOBWs, $\psi_{n,m}^{\gamma}(t) := \psi(k,n,m,\gamma,t)$ have five arguments, k is the level of resolution and can be any positive integer, $n=1,\dots,2^{k-1}$, m is the degree of Bernoulli polynomials, γ is a real positive number and t denotes time. These functions are defined on the interval $[0,T)$ as

$$ \psi_{n,m}^{\gamma}(t) = \begin{cases} 2^{\frac{k-1}{2}}B_m\left(\frac{2^{k-1}}{T^{\gamma}}t^{\gamma}-n+1\right), & \left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T \leq t < \left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}}T, \\ 0, & \text{Otherwise}, \end{cases} \tag{3.3} $$

where $B_m(t) = \delta_m\mathfrak{b}_m(t)$, with δ_m as the normality factor defined as

$$ \delta_m = \begin{cases} \sqrt{\frac{\gamma}{T^{\gamma}}}, & m=0, \\ \dfrac{\sqrt{\frac{\gamma}{T^{\gamma}}}}{\sqrt{\frac{(-1)^{m-1}(m!)^2b_{2m}}{(2m)!}}}, & m>0. \end{cases} $$

A function $y \in L^2[0,T)$ can be expanded based on the GFOBWs as:

$$y(t) \simeq y_{k,M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} y_{n,m}\psi_{n,m}^{\gamma}(t) = Y^T \Psi^{\gamma}(t), \tag{3.4}$$

where Y and $\Psi^{\gamma}(t)$ are the coefficient vector and basis functions vector, respectively, given by

$$Y = \left[y_{1,0}, y_{1,1}, \ldots, y_{1,M}, \ldots, y_{2^{k-1},0}, y_{2^{k-1},1}, \ldots, y_{2^{k-1},M}\right]^T,$$

$$\Psi^{\gamma}(t) = \left[\psi_{1,0}^{\gamma}(t), \ldots, \psi_{1,M}^{\gamma}(t), \ldots, \psi_{2^{k-1},0}^{\gamma}(t), \ldots, \psi_{2^{k-1},M}^{\gamma}(t)\right]^T. \tag{3.5}$$

The coefficient vector Y can be computed by the following formula

$$Y^T = \hat{Y}^T Q^{-1},$$

where Q is given as

$$Q = \langle \Psi^{\gamma}, \Psi^{\gamma} \rangle = \int_0^T t^{\gamma-1} \Psi^{\gamma}(t) \Psi^{\gamma T}(t) dt, \tag{3.6}$$

and

$$\hat{Y} = \left[\hat{y}_{1,0}, \hat{y}_{1,1}, \ldots, \hat{y}_{1,M}, \ldots, \hat{y}_{2^{k-1},0}, \hat{y}_{2^{k-1},1}, \ldots, \hat{y}_{2^{k-1},M}\right]^T,$$

with

$$\hat{y}_{n,m} = \langle y, \psi_{n,m}^{\gamma} \rangle = \int_0^T t^{\gamma-1} y(t) \psi_{n,m}^{\gamma}(t) dt.$$

It should be noted that according to the properties of the GFOBWs, the matrix Q is a symmetric positive definite matrix with all principal diagonal entries equal to 1. For example, with $k = 2$, $M = 4$, $\gamma = \frac{1}{2}$ and $T = 2$, we have

$$Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\sqrt{\frac{7}{10}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -2\sqrt{\frac{5}{21}} & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{\frac{7}{10}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\sqrt{\frac{5}{21}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\sqrt{\frac{7}{10}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2\sqrt{\frac{5}{21}} \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{7}{10}} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\sqrt{\frac{5}{21}} & 0 & 1
\end{bmatrix}.$$

Remark 3.3. Using property (3.2) and the definition of the GFOBWs in (3.3), it can be easily shown that by considering $M \leq 2$ and with every positive integer number k and every positive real number γ, the matrix Q given by (3.6) is equal to the identity matrix of dimension $2^{k-1}(M+1) \times 2^{k-1}(M+1)$. For $M \geq 3$, the condition number of Q becomes larger and larger as M increases. This makes the problem of approximating a given function in terms of the Bernoulli polynomials and accordingly in terms of the GFOBWs an ill-posed problem. A way to overcome this problem is to choose the value of M less than 3 and increasing the value of k to get more accurate approximations.

4. Numerical method

This section is concerned on presenting a new numerical technique based on the GFOBWs for solving problem (1.1)-(1.2). To this aim, we rewrite Equation (1.1) as follows:

$$ {}_0^C D_t^{\alpha} y(t) = f(t, y(t)) + \frac{\Gamma(1-\beta)}{\Gamma(1-\beta)} \int_0^t (t-s)^{(1-\beta)-1} g(s, y(s)) ds. $$

By utilizing the definition of Riemann-Liouville integral in the above equation, it becomes

$$ {}_0^C D_t^{\alpha} y(t) = f(t, y(t)) + \Gamma(1-\beta) {}_0 I_t^{1-\beta} h(t), \tag{4.1} $$

where we have considered

$$ h(t) = g(t, y(t)). \tag{4.2} $$

Suppose that

$$ {}_0^C D_t^{\alpha} y(t) \simeq Y^T \Psi^{\gamma}(t), \tag{4.3} $$

$$ h(t) \simeq H^T \Psi^{\gamma}(t). \tag{4.4} $$

Then, using the initial conditions given by (1.2) and property (2.2), we obtain

$$ y(t) = Y^T ({}_0 I_t^{\alpha} \Psi^{\gamma}(t)) + s(t), \tag{4.5} $$

with

$$ s(t) = \sum_{i=0}^{\lceil \alpha \rceil - 1} y_i \frac{t^i}{i!}. $$

Now, we need to compute the Riemann-Liouville fractional integral of the vector $\Psi^{\gamma}(t)$ given by (3.5). We define

$$ {}_0 I_t^{\alpha} \Psi^{\gamma}(t) = \overline{\Psi}^{\gamma}(t, \alpha), \tag{4.6} $$

where

$$ \overline{\Psi}^{\gamma}(t, \alpha) = \left[{}_0 I_t^{\alpha} \psi_{1,0}^{\gamma}(t), \ldots, {}_0 I_t^{\alpha} \psi_{1,M}^{\gamma}(t), \ldots, {}_0 I_t^{\alpha} \psi_{2^{k-1},0}^{\gamma}(t), \ldots, {}_0 I_t^{\alpha} \psi_{2^{k-1},M}^{\gamma}(t) \right]^T. $$

To compute ${}_0 I_t^{\alpha} \psi_{n,m}^{\gamma}(t)$ ($n = 1, \ldots, 2^{k-1}$, $m = 0, \ldots, M$), first, we present the function $\psi_{n,m}^{\gamma}$, defined in (3.3), in the following form

$$ \psi_{n,m}^{\gamma}(t) = 2^{\frac{k-1}{2}} \delta_m \left(u_{\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}} T}(t) - u_{\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}} T}(t) \right) \mathfrak{b}_m \left(\frac{2^{k-1}}{T^{\gamma}} t^{\gamma} - n + 1 \right), $$

where u_a is defined as

$$ u_a(t) = \begin{cases} 1, & t \geq a, \\ 0, & t < a. \end{cases} $$

Taking (3.1) into account, we get

$$ \psi_{n,m}^{\gamma}(t) = 2^{\frac{k-1}{2}} \delta_m \left(u_{\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}} T}(t) - u_{\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}} T}(t) \right) \sum_{j=0}^{m} \binom{m}{j} b_{m-j} \left(\frac{2^{k-1}}{T^{\gamma}} t^{\gamma} - n + 1 \right)^j. \tag{4.7} $$

Using the binomial theorem in (4.7), we have

$$ \psi_{n,m}^{\gamma}(t) = 2^{\frac{k-1}{2}} \delta_m \sum_{j=0}^{m} \sum_{r=0}^{j} \binom{m}{j} \binom{j}{r} b_{m-j} \left(\frac{2^{k-1}}{T^{\gamma}} \right)^r (1-n)^{j-r} t^{\gamma r} \left(u_{\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}} T}(t) - u_{\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}} T}(t) \right). \tag{4.8} $$

Applying the Riemann-Liouville integral operator of order α to both sides of (4.8) gives

$$
\begin{aligned}
{}_0I_t^{\alpha}\psi_{n,m}^{\gamma}(t) &= 2^{\frac{k-1}{2}}\delta_m \sum_{j=0}^{m}\sum_{r=0}^{j}\binom{m}{j}\binom{j}{r} b_{m-j}\left(\frac{2^{k-1}}{T^{\gamma}}\right)^{r}(1-n)^{j-r} \\
&\times\left({}_0I_t^{\alpha}\left(t^{\gamma r}u_{\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T}(t)\right)-{}_0I_t^{\alpha}\left(t^{\gamma r}u_{\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}}T}(t)\right)\right).
\end{aligned}
\tag{4.9}
$$

Let $\alpha,\nu,a\geq 0$ be real numbers. Taking property (2.1) into account yields

$$
\begin{aligned}
{}_0I_t^{\alpha}(t^{\nu}u_a(t)) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\nu}u_a(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\nu}ds-\frac{1}{\Gamma(\alpha)}\int_0^a (t-s)^{\alpha-1}s^{\nu}ds \\
&= {}_0I_t^{\alpha}(t^{\nu})-\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^a s^{\nu}(1-\frac{s}{t})^{\alpha-1}ds \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}t^{\nu+\alpha}-\frac{t^{\nu+\alpha}}{\Gamma(\alpha)}\beta\left(\frac{a}{t};\nu+1,\alpha\right) \\
&= \left(\frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)}-\frac{\beta\left(\frac{a}{t};\nu+1,\alpha\right)}{\Gamma(\alpha)}\right)t^{\nu+\alpha}.
\end{aligned}
\tag{4.10}
$$

In the above result, $\beta(z;a,b)$ denotes the incomplete beta function given by

$$
\beta(z;a,b)=\int_0^z s^{a-1}(1-s)^{b-1}ds.
$$

Substituting (4.10) into (4.9) yields

$$
{}_0I_t^{\alpha}\psi_{n,m}^{\gamma}(t)=\begin{cases}0, & t<\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T,\\ \Omega_1(t), & \left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T\leq t<\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}}T,\\ \Omega_2(t), & t\geq\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}}T,\end{cases}
$$

where

$$
\begin{aligned}
\Omega_1(t) &= 2^{\frac{k-1}{2}}\delta_m \sum_{j=0}^{m}\sum_{r=0}^{j}\binom{m}{j}\binom{j}{r} b_{m-j}\left(\frac{2^{k-1}}{T^{\gamma}}\right)^{r}(1-n)^{j-r} \\
&\times\left(\frac{\Gamma(\gamma r+1)}{\Gamma(\gamma r+\alpha+1)}-\frac{\beta\left(\frac{\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T}{t};\gamma r+1,\alpha\right)}{\Gamma(\alpha)}\right)t^{\gamma r+\alpha},
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2(t) &= 2^{\frac{k-1}{2}}\delta_m \sum_{j=0}^{m}\sum_{r=0}^{j}\binom{m}{j}\binom{j}{r} b_{m-j}\left(\frac{2^{k-1}}{T^{\gamma}}\right)^{r}(1-n)^{j-r} \\
&\times\left(\frac{\beta\left(\frac{\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}}T}{t};\gamma r+1,\alpha\right)}{\Gamma(\alpha)}-\frac{\beta\left(\frac{\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T}{t};\gamma r+1,\alpha\right)}{\Gamma(\alpha)}\right)t^{\gamma r+\alpha}.
\end{aligned}
$$

Having computed ${}_0I_t^{\alpha}\psi_{n,m}^{\gamma}(t)$, by substituting (4.6) into (4.5), an approximation of $y(t)$ is given by

$$y(t) \simeq Y^T\overline{\Psi}^{\gamma}(t,\alpha) + s(t). \tag{4.11}$$

Substituting approximations (4.3), (4.4), and (4.11) into (4.1) yields

$$Y^T\Psi^{\gamma}(t) = f\left(t, Y^T\overline{\Psi}^{\gamma}(t,\alpha) + s(t)\right) + \Gamma(1-\beta)H^T\overline{\Psi}^{\gamma}(t, 1-\beta). \tag{4.12}$$

The remainder corresponding to Equation (4.12) is defined as

$$R_1(t) = Y^T\Psi^{\gamma}(t) - f\left(t, Y^T\overline{\Psi}^{\gamma}(t,\alpha) + s(t)\right) - \Gamma(1-\beta)H^T\overline{\Psi}^{\gamma}(t, 1-\beta). \tag{4.13}$$

Moreover, using approximations (4.4) and (4.11) in (4.2), we define the second remainder function as

$$R_2(t) = H^T\Psi^{\gamma}(t) - g\left(t, Y^T\overline{\Psi}^{\gamma}(t,\alpha) + s(t)\right). \tag{4.14}$$

Next, we define the following collocation points

$$t_{n,m} = T\left[\frac{1}{2^k}\left(\cos\left(\frac{(2m+1)\pi}{2(M+1)}\right) + 2n - 1\right)\right]^{\frac{1}{\gamma}}, \; n = 1, \cdots, 2^{k-1}, \; m = 0, \cdots, M, \tag{4.15}$$

which are the shifted Chebyshev nodes distributed in the interval $\left(\left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}}T, \left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}}T\right)$ for each fixed n. Finally, we set the remainders defined in (4.13) and (4.14) at the collocation points $t_{n,m}$, $n = 1, \cdots, 2^{k-1}$, $m = 0, \cdots, M$, equal to zero to obtain a system of $2^k(M+1)$ algebraic equations as follows:

$$R_1(t_{n,m}) = 0,$$
$$R_2(t_{n,m}) = 0.$$

By solving this system and finding the elements of the vector Y, the numerical solution of problem (1.1)-(1.2) is computed by (4.11).

Remark 4.1. In the case of linear fractional integro-differential equations, the remainder function $R(t)$ to determine the elements of Y is given by

$$R(t) = Y^T\Psi^{\gamma}(t) - f\left(t, Y^T\overline{\Psi}^{\gamma}(t,\alpha) + s(t)\right) - \Gamma(1-\beta)\left(Y^T\overline{\Psi}^{\gamma}(t, 1-\beta+\alpha) + \sum_{i=0}^{\lceil\alpha\rceil-1} y_i \frac{t^{i+1-\beta}}{\Gamma(i+2-\beta)}\right).$$

Now, we extend the above suggested method for solving Equation (1.3). To do this, first, using the change of variable $s = \frac{T}{2}(\tau+1)$, we transform the interval $[0,T]$ to the interval $[-1,1]$ and obtain

$${}_0^CD_t^{\alpha}y(t) = f(t, y(t)) + \int_0^t (t-s)^{-\beta} g(s, y(s))\, ds + \frac{T}{2}\int_{-1}^{1} \kappa\left(t, \frac{T}{2}(\tau+1), y\left(\frac{T}{2}(\tau+1)\right)\right) d\tau.$$

Then, using the Gauss-Legendre quadrature formula, we get

$${}_0^CD_t^{\alpha}y(t) \simeq f(t, y(t)) + \int_0^t (t-s)^{-\beta} g(s, y(s))\, ds + \frac{T}{2}\sum_{l=1}^{N} \kappa\left(t, \frac{T}{2}(\tau_l+1), y\left(\frac{T}{2}(\tau_l+1)\right)\right),$$

where τ_l, $l = 1, \ldots, N$, are the zeros of Legendre polynomial of degree N and w_l are the corresponding weights [5]. By following a similar procedure introduced for solving problem (1.1)-(1.2) and considering the same remainder function R_2, we define R_1 as follows:

$$\begin{aligned} R_1(t) &= Y^T\Psi^{\gamma}(t) - f\left(t, Y^T\overline{\Psi}^{\gamma}(t,\alpha) + s(t)\right) - \Gamma(1-\beta)H^T\overline{\Psi}^{\gamma}(t, 1-\beta) \\ &- \frac{T}{2}\sum_{l=1}^{N} \kappa\left(t, \frac{T}{2}(\tau_l+1), Y^T\overline{\Psi}^{\gamma}\left(\frac{T}{2}(\tau_l+1), \alpha\right) + s\left(\frac{T}{2}(\tau_l+1)\right)\right). \end{aligned}$$

At last, we set the remainders R_1 and R_2 equal to zero at the collocation points given by (4.15), and find the approximate solution by solving the obtained system.

5. Error estimate

This section focuses on presenting an estimation of the error for the approximation of a given function $y \in L^2[0,T)$ based on the GFOBWs. To this aim, we define

$$w^{\gamma}(t) = t^{\gamma-1},$$
$$I_{n,\gamma} = \left[t_{n-1}^{k,\gamma}, t_n^{k,\gamma}\right), \quad n = 1, \dots, 2^{k-1},$$

where

$$t_n^{k,\gamma} = \left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}} T, \qquad n = 0, 1, \dots, 2^{k-1}.$$

Let $P_M(I_{n,\gamma})$, $n = 1, 2, \dots, 2^{k-1}$, be the space of all polynomials of degree less than or equal to M on the subinterval $I_{n,\gamma}$. Also, suppose that $L^2[0,T)$ and $L^2(I_{n,\gamma})$ are the spaces of all measurable functions that their square is Lebesgue integrable with respect to the weight function w^{γ} on $[0,T)$ and $I_{n,\gamma}$, respectively. These spaces are equipped with the following norms

$$\|y\|_{L^2[0,T)} = \left(\int_0^T w^{\gamma}(t)|y(t)|^2 dt\right)^{\frac{1}{2}},$$

$$\|y\|_{L^2(I_{n,\gamma})} = \left(\int_{t_{n-1}^{k,\gamma}}^{t_n^{k,\gamma}} w^{\gamma}(t)|y(t)|^2 dt\right)^{\frac{1}{2}}.$$

If y is a sufficiently smooth function on the subinterval $I_{n,\gamma}$ and $p_{n,M}$ is its interpolant at $(M+1)$ points $t_{n,m}$, $m = 0, 1, \dots, M$, defined by (4.15), then

$$y(t) - p_{n,M}(t) = \frac{y^{(M+1)}(\xi)}{(M+1)!} \prod_{m=0}^{M} (t - t_{n,m}), \qquad \xi \in I_{n,\gamma}.$$

Consequently,

$$\begin{aligned} |y(t) - p_{n,M}(t)| &\le \frac{1}{(M+1)!} \max_{t \in I_{n,\gamma}} |y^{(M+1)}(t)| \max_{t \in I_{n,\gamma}} \prod_{m=0}^{M} |t - t_{n,m}| \\ &\le \frac{1}{(M+1)!} \max_{t \in I_{n,\gamma}} |y^{(M+1)}(t)| \left(\left(\frac{n}{2^{k-1}}\right)^{\frac{1}{\gamma}} T - \left(\frac{n-1}{2^{k-1}}\right)^{\frac{1}{\gamma}} T\right)^{M+1} \\ &= \frac{T^{M+1}}{(M+1)!} 2^{-\left(\frac{k-1}{\gamma}\right)(M+1)} \max_{t \in I_{n,\gamma}} |y^{(M+1)}(t)| \sum_{i=0}^{M+1} (-1)^i \binom{M+1}{i} n^{\frac{M+1-i}{\gamma}} (n-1)^{\frac{i}{\gamma}}. \end{aligned} \tag{5.1}$$

The above result helps us to prove the following lemma.

Lemma 5.1. *Let k be a positive integer and $y_n : I_{n,\gamma} \longrightarrow \mathbb{R}$, $n = 1, \dots, 2^{k-1}$, be a sufficiently smooth function. Suppose that $\tilde{y}_n(t) = \sum_{m=0}^{M} y_{n,m} \psi_{n,m}^{\gamma}(t)$ denotes the approximation of y_n based on the GFOBWs. Then*

$$\|y_n - \tilde{y}_n\|_{L^2(I_{n,\gamma})} \le \frac{T^{M+1+\frac{\gamma}{2}}}{\sqrt{\gamma}(M+1)!} \Lambda_{n,M} 2^{-(k-1)\left(\frac{M+1}{\gamma}+\frac{1}{2}\right)} \sum_{i=0}^{M+1} (-1)^i \binom{M+1}{i} n^{\frac{M+1-i}{\gamma}} (n-1)^{\frac{i}{\gamma}},$$

where

$$\Lambda_{n,M} = \max_{t \in I_{n,\gamma}} |y_n^{(M+1)}(t)|.$$

Proof. We know that $\tilde{y}_n$ is the best approximation of y_n in $P_M(I_{n,\gamma})$. Therefore

$$\|y_n - \tilde{y}_n\|_{L^2(I_{n,\gamma})} \leq \|y_n - q\|_{L^2(I_{n,\gamma})},$$

where q is any arbitrary polynomial in $P_M(I_{n,\gamma})$. Hence for $p_{n,M}$ as the interpolating polynomial of y_n at $(M+1)$ points $t_{n,m}$, defined by (4.15), we write

$$\|y_n - \tilde{y}_n\|_{L^2(I_{n,\gamma})} \leq \|y_n - p_{n,M}\|_{L^2(I_{n,\gamma})}. \tag{5.2}$$

Using (5.1) and (5.2), we obtain

$$\begin{aligned}
\|y_n - \tilde{y}_n\|^2_{L^2(I_{n,\gamma})} &\leq \int_{t^{k,\gamma}_{n-1}}^{t^{k,\gamma}_n} w^\gamma(t)|y_n - p_{n,M}|^2 dt \\
&\leq \left(\frac{T^{M+1}}{(M+1)!}\Lambda_{n,M}2^{-(\frac{k-1}{\gamma})(M+1)}\sum_{i=0}^{M+1}(-1)^i\binom{M+1}{i}n^{\frac{M+1-i}{\gamma}}(n-1)^{\frac{i}{\gamma}}\right)^2 \times \int_{t^{k,\gamma}_{n-1}}^{t^{k,\gamma}_n} t^{\gamma-1}dt \\
&= \left(\frac{T^{M+1}}{(M+1)!}\Lambda_{n,M}2^{-(\frac{k-1}{\gamma})(M+1)}\sum_{i=0}^{M+1}(-1)^i\binom{M+1}{i}n^{\frac{M+1-i}{\gamma}}(n-1)^{\frac{i}{\gamma}}\right)^2 \frac{T^\gamma}{\gamma 2^{k-1}},
\end{aligned}$$

which leads to the desired result. ☐

Theorem 5.2. *Let y be a continuous function defined on $[0,T)$ that has sufficiently smooth restrictions on each subinterval $I_{n,\gamma}$. Assume that $y_n : I_{n,\gamma} \longrightarrow \mathbb{R}$, $n = 1, \ldots, 2^{k-1}$, be defined by $y_n(t) = y(t)$ for all $t \in I_{n,\gamma}$. If $y_{k,M}$, given by (3.4), is the approximation of y in terms of the GFOBWs, then*

$$\|y - y_{k,M}\|_{L^2[0,T)} \leq \frac{T^{M+1+\frac{\gamma}{2}}}{(M+1)!}\Lambda_M 2^{-(k-1)(\frac{M+1}{\gamma}+\frac{1}{2})}\sqrt{\frac{\Theta_{k,M}}{\gamma}}, \tag{5.3}$$

where $\Lambda_M = \max_{1\leq n\leq 2^{k-1}} \Lambda_{n,M}$ and

$$\Theta_{k,M} = \sum_{n=1}^{2^{k-1}}\left(\sum_{i=0}^{M+1}(-1)^i\binom{M+1}{i}n^{\frac{M+1-i}{\gamma}}(n-1)^{\frac{i}{\gamma}}\right)^2.$$

Proof. Using the notations introduced in Lemma 5.1, we have

$$\begin{aligned}
\|y - y_{k,M}\|^2_{L^2[0,T)} &= \int_0^T w^\gamma(t)|y(t) - y_{k,M}(t)|^2 dt \\
&= \sum_{n=1}^{2^{k-1}} \int_{t^{k,\gamma}_{n-1}}^{t^{k,\gamma}_n} w^\gamma(t)|y_n(t) - \tilde{y}_n(t)|^2 dt \\
&= \sum_{n=1}^{2^{k-1}} \|y_n - \tilde{y}_n\|^2_{L^2(I_{n,\gamma})} \\
&\leq \sum_{n=1}^{2^{k-1}} \left(\frac{T^{M+1+\frac{\gamma}{2}}}{\sqrt{\gamma}(M+1)!}\Lambda_{n,M}2^{-(k-1)(\frac{M+1}{\gamma}+\frac{1}{2})}\sum_{i=0}^{M+1}(-1)^i\binom{M+1}{i}n^{\frac{M+1-i}{\gamma}}(n-1)^{\frac{i}{\gamma}}\right)^2 \\
&\leq \left(\frac{T^{M+1+\frac{\gamma}{2}}}{\sqrt{\gamma}(M+1)!}\Lambda_M 2^{-(k-1)(\frac{M+1}{\gamma}+\frac{1}{2})}\right)^2 \times \sum_{n=1}^{2^{k-1}}\left(\sum_{i=0}^{M+1}(-1)^i\binom{M+1}{i}n^{\frac{M+1-i}{\gamma}}(n-1)^{\frac{i}{\gamma}}\right)^2,
\end{aligned}$$

which gives (5.3) by taking square root. ☐

Remark 5.3. In many cases, solutions of fractional integro-differential equations suffer lack of regularity at origin. Since no information is available on the analytical results, presenting an error discussion for the approximate solution given by the proposed method is difficult. However, according to the numerical results obtained in the next section, choosing a small value of M and increasing the level of resolution in the implementation of the method can be a suggestion to give more accurate results in the case of equations with non-smooth solutions.

6. Numerical examples

In this section, some examples are solved using the proposed method to illustrate its accuracy and efficiency. Let y and $\hat{y}^{\gamma}_{k,M}$ be, respectively, the exact and approximate solutions in each example. We define the following notations to check the error of the method

$$\varepsilon^{\gamma}_{k,M}(t) = |y(t) - \hat{y}^{\gamma}_{k,M}(t)|,$$
$$\zeta^{\gamma}_{k,M} = \max_{t\in[0,T]} \varepsilon^{\gamma}_{k,M}(t).$$

For our simulations, we used Mathematica software and in the computation of the Gauss-Legendre quadrature, we have used $N = 8$.

Example 6.1. As a first example, consider Equation (1.1) with [3, 11]

$$\alpha = \frac{2}{3}, \quad \beta = \frac{1}{2}, \quad f(t,y) = \frac{3\Gamma(\frac{1}{2})}{4\Gamma(\frac{11}{6})}t^{\frac{5}{6}} - t^{\frac{5}{2}} - \frac{32}{35}t^{\frac{7}{2}} + ty, \quad g(s,y) = y^2, \quad T = 1.$$

By having the initial condition $y(0) = 0$, the exact solution of this problem is $y(t) = t^{\frac{3}{2}}$.

We have implemented the proposed method for solving this problem with different values of the parameters γ, k and M to check the effect of each of these parameters in the accuracy of the method. Table 1 reports the absolute error at some points obtained by the present method with $k = 5$, $M = 3$ and $\gamma = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$, method of [3] based on the Taylor wavelet functions and also method of [11] based on the quadratic hat functions. These results show that almost using a same number of basis functions, the present method gives more accurate results with $\gamma = \frac{5}{6}$. We note that the values of γ used in the implementation of our method have been chosen taking into account the degrees of the power functions existing in the function f and the values of α and β. Moreover, plots of the absolute error function $\varepsilon^{\gamma}_{k,M}(t)$ are seen in Figure 1. Since we obtained the optimal results with $\gamma = \frac{5}{6}$, the method has been employed with this value of γ and different values of k and M. First, we choose $M = 4$ and increase the value of k and then by choosing $k = 1$, we increase the value of M. The numerical results of the maximum absolute error $\zeta^{\gamma}_{k,M}$ in a logarithmic scale can be seen in Figure 2. By observing these results, one can find out the error decreases as the value of k increases, but, as it was mentioned in Remark 3.3, the method becomes unstable for large values of M.

Table 1. Comparison of the absolute error at some points for Example 6.1.

t	Present method ($k=5$, $M=3$)					Method of [3]	Method of [11]
	$\gamma=\frac{1}{3}$	$\gamma=\frac{1}{2}$	$\gamma=\frac{2}{3}$	$\gamma=\frac{5}{6}$	$\gamma=1$	$r=5$, $S=4$	$n=64$
0.125	4.31e−10	5.84e−9	1.28e−7	7.01e−10	3.90e−6	5.22e−7	1.70e−5
0.250	5.75e−9	9.53e−9	1.13e−7	2.03e−9	3.27e−6	1.21e−7	1.49e−5
0.375	5.21e−8	1.31e−8	1.16e−7	3.37e−9	3.27e−6	4.83e−8	1.51e−5
0.500	6.81e−8	3.78e−8	1.31e−7	5.10e−9	3.66e−6	2.67e−8	1.69e−5
0.625	1.68e−7	5.24e−8	1.65e−7	7.50e−9	4.51e−6	1.45e−8	2.09e−5
0.750	4.84e−7	7.81e−8	2.31e−7	1.20e−8	6.12e−6	9.35e−9	2.83e−5
0.875	9.62e−7	1.52e−7	3.49e−7	1.93e−8	9.12e−6	6.57e−9	4.22e−5

Example 6.2. Consider problem (1.1)-(1.2) with [3, 11]:

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{2}, \quad f(t,y) = \frac{6}{\Gamma(\frac{11}{3})}t^{\frac{8}{3}} + \left(\frac{32}{35} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)}\right)t^{\frac{11}{6}} + \Gamma\left(\frac{7}{3}\right)t - \frac{32}{35}t^{\frac{1}{2}}y,$$
$$g(s,y) = y,$$

under the initial condition $y(0) = 0$. The exact solution is $y(t) = t^3 + t^{\frac{4}{3}}$.

Similar to the first example, we first consider $T = 1$, $k = 5$, $M = 3$, and $\gamma = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1$, and compared the obtained results with the results of quadratic hat functions method [11] and Taylor wavelet functions method [3] in

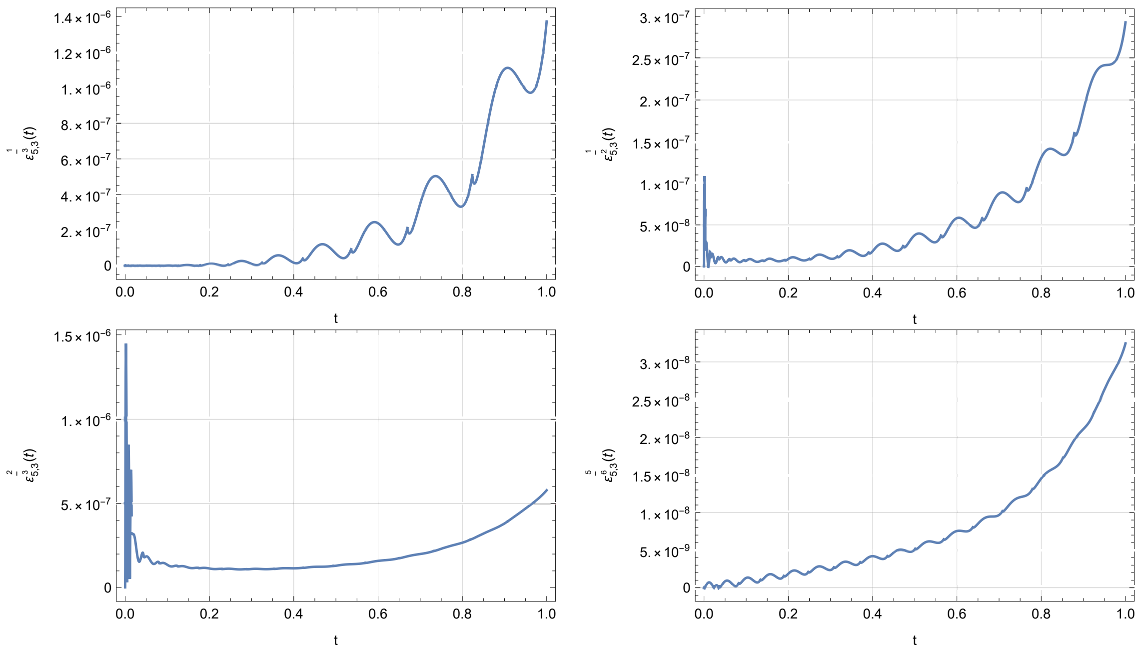

FIGURE 1. (Example 6.1.) Plots of the absolute error function $\varepsilon^{\gamma}_{k,M}(t)$ with $k = 5$, $M = 3$ and $\gamma = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$.

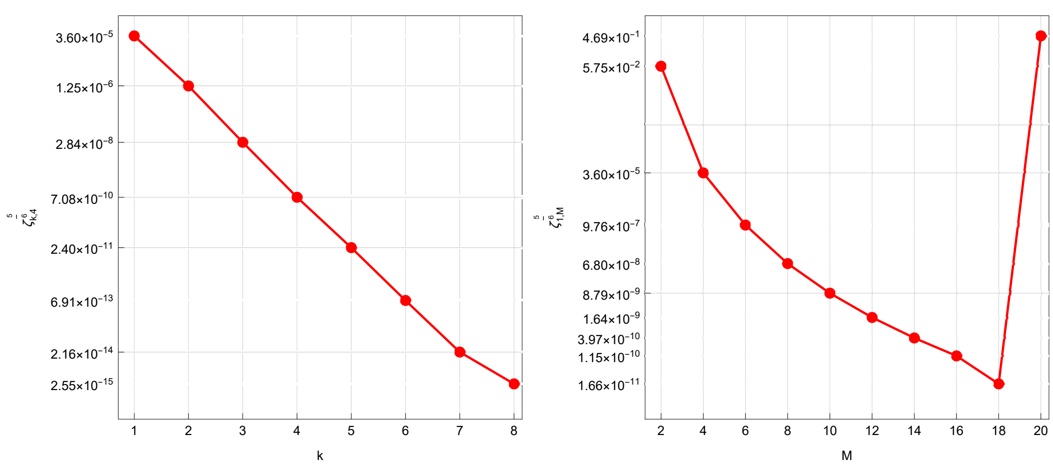

FIGURE 2. (Example 6.1.) Numerical results of the maximum absolute error $\zeta^{\gamma}_{k,M}$ in logarithmic scale with: $M = 4$, $\gamma = \frac{5}{6}$ and different values of k (left), $k = 1$, $\gamma = \frac{5}{6}$ and different values of M (right).

Table 2. Moreover, the absolute error function $\varepsilon^{\gamma}_{k,M}(t)$ is plotted in Figure 3. As it is seen from these results, among all the tested values of γ, we obtain the best results with $\gamma = 1$. The results obtained for $\zeta^{\gamma}_{k,M}$ with this value of γ and different values of k and M are displayed in Figure 4. From this figure, one can see that by increasing the value of k, the maximum absolute error decreases (left) and the method is not stable for large M (right). Furthermore, to check the efficiency of the method for large intervals, we set $T = 10$, $M = 1$, $\gamma = 1$ and increased the value of k from 1 to 4. Plots of the approximate solutions together with the exact one are shown in Figure 5. It is evident that the numerical solutions converge to exact solution as k increases.

TABLE 2. Comparison of the absolute error at some points for Example 6.2.

t	Present method ($k=5,\ M=3$)					Method of [3]	Method of [11]
	$\gamma=\frac{1}{3}$	$\gamma=\frac{1}{2}$	$\gamma=\frac{2}{3}$	$\gamma=\frac{5}{6}$	$\gamma=1$	$r=5,\ S=4$	$n=64$
0.125	2.51e−8	3.60e−8	8.08e−8	6.96e−7	5.73e−9	6.91e−7	2.40e−6
0.250	2.13e−7	1.66e−8	9.84e−8	5.80e−7	1.10e−8	1.47e−7	2.43e−6
0.375	6.69e−7	8.02e−8	1.11e−7	5.91e−7	1.28e−8	6.25e−8	2.58e−6
0.500	3.54e−7	1.21e−7	1.17e−7	6.21e−7	1.42e−8	3.82e−8	2.76e−6
0.625	7.12e−7	1.20e−7	9.21e−8	6.66e−7	1.56e−8	2.91e−8	2.99e−6
0.750	1.89e−6	2.63e−7	1.59e−7	7.09e−7	1.71e−8	2.55e−8	3.22e−6
0.875	1.97e−6	1.72e−7	9.69e−8	7.69e−7	1.85e−8	2.41e−8	3.48e−6

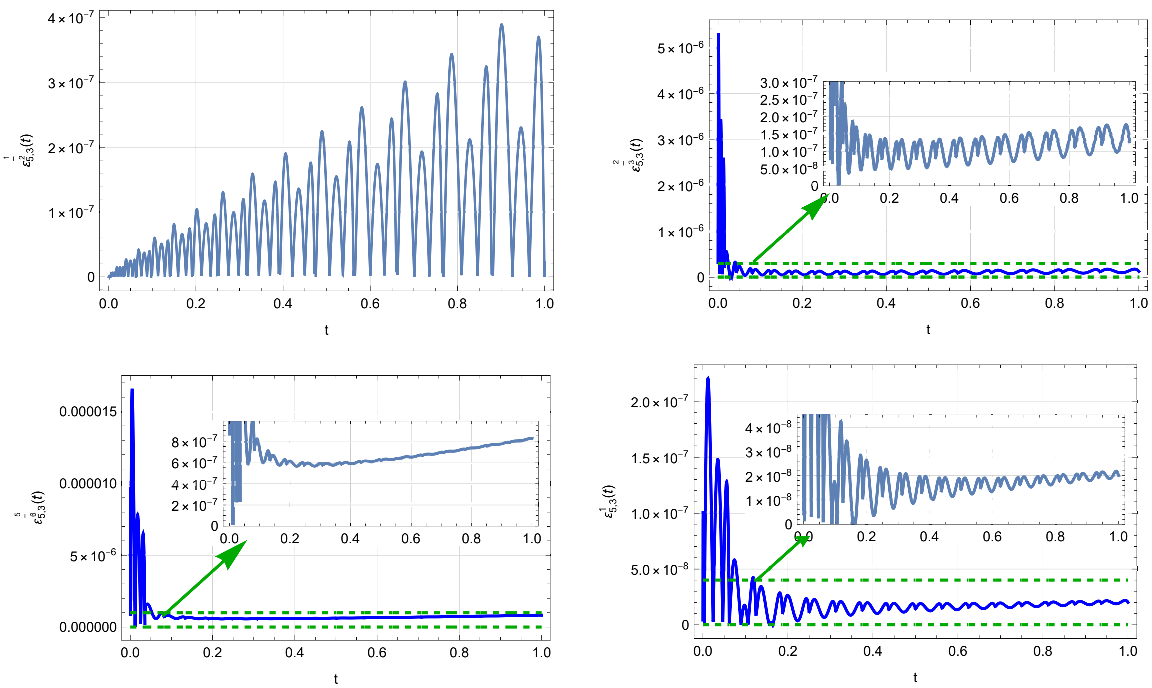

FIGURE 3. (Example 6.2.) Plots of the absolute error function $\varepsilon^{\gamma}_{k,M}(t)$ with $k=5$, $M=3$ and $\gamma=\frac{1}{2},\ \frac{2}{3},\ \frac{5}{6},\ 1$.

Example 6.3. Consider problem (1.1)-(1.2) with [3, 12]:

$$0<\alpha\le 1,\quad \beta=\frac{1}{2},\quad f(t,y)=2t-\frac{16}{15}t^{\frac{1}{2}}y,\quad g(s,y)=y,\quad T=1,\quad y(0)=0,$$

which has the exact solution $y(t)=t^2$ when $\alpha=1$.

In the implementation of our method, by considering $k=1$, $M=1$, $\gamma=1$ and $\alpha=1$, we set:

$${}^{C}_{0}D^{1}_{t}y(t)=Y^T\Psi^1(t),$$

where

$$Y=[y_{1,0},y_{1,1}]^T,\quad \Psi^1(t)=[1,\sqrt{3}(2t-1)]^T.$$

Taking the initial condition into account, we get

$$y(t)=Y^T\overline{\Psi}^1(t,1),$$

with

$$\overline{\Psi}^1(t,1)=[t,\sqrt{3}(t^2-t)]^T.$$

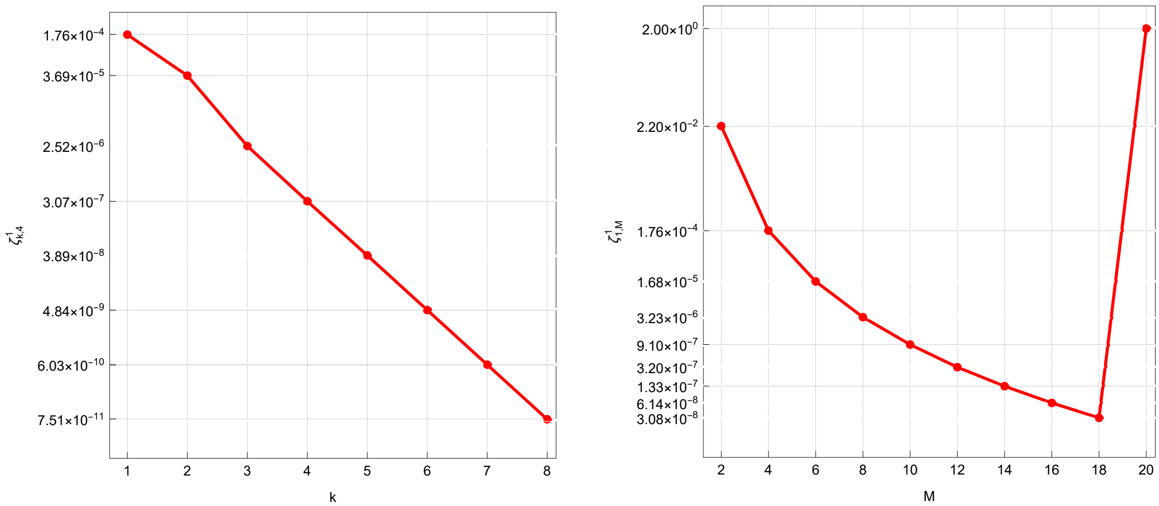

FIGURE 4. (Example 6.2.) Numerical results of the maximum absolute error $\zeta^{\gamma}_{k,M}$ in logarithmic scale with: $M = 4$, $\gamma = 1$ and different values of k (left), $k = 1$, $\gamma = 1$ and different values of M (right).

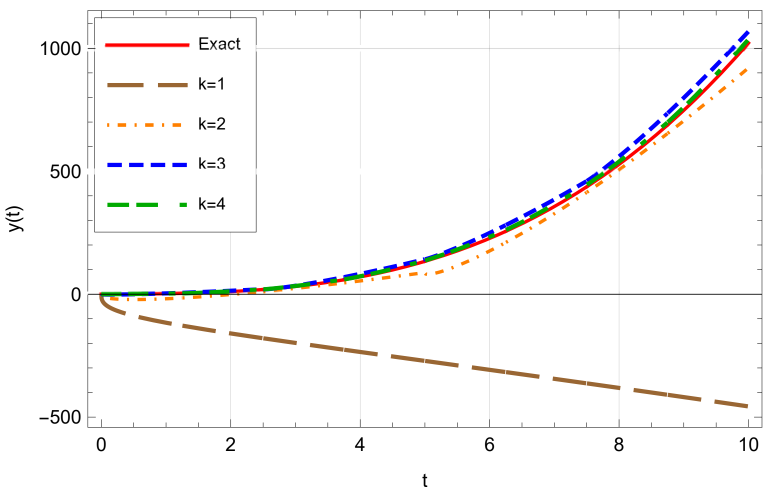

FIGURE 5. (Example 6.2.) Plots of the exact solution together with the approximate solutions obtained by $M = 1$, $\gamma = 1$ and $k = 1, 2, 3, 4$ on the interval $[0, 10]$.

We also need to compute $\overline{\Psi}^1\left(t, \frac{3}{2}\right)$ to obtain the remainder function. This vector is given by

$$\overline{\Psi}^1\left(t, \frac{3}{2}\right) = \left[\frac{4}{3\sqrt{\pi}} t^{3/2}, \frac{4}{5\sqrt{3\pi}} t^{3/2}(4t-5)\right]^T.$$

By utilizing the aforementioned approximations in the main problem, we have

$$R(t) = \left(1 - \frac{4t^{3/2}}{15}\right) y_{1,0} + \frac{\left(4t^{3/2} + 30t - 15\right)}{5\sqrt{3}} y_{1,1} - 2t = 0.$$

To give the final system, we use the following collocation points into this equation

$$t_{1,0} = \frac{2+\sqrt{2}}{4}, \qquad t_{1,1} = \frac{2-\sqrt{2}}{4}.$$

After solving the obtained system, we have

$$y_{1,0} = 1, \qquad y_{1,1} = \frac{\sqrt{3}}{3},$$

which gives the exact solution.

To check the effect of the parameter γ, the method has been applied to solve this problem with $k = 1$, $M = 2$, $\alpha = 1$ and $\gamma = 0.1, 0.2, 0.3, 0.4, 0.5$ and the results have been displayed in Figure 6 (left). We note here that by testing different values of γ, it is seen the method gives the exact solution with every $\gamma \geq 0.5$ and only three basis functions. We also implemented the method with $k = 2$, $M = 2$, $\gamma = 1$ and different values of α. The numerical solutions are plotted in Figure 6 (right) along with the exact solution of the case $\alpha = 1$. These results show that the numerical solutions converge to the exact one as $\alpha \longrightarrow 1$.

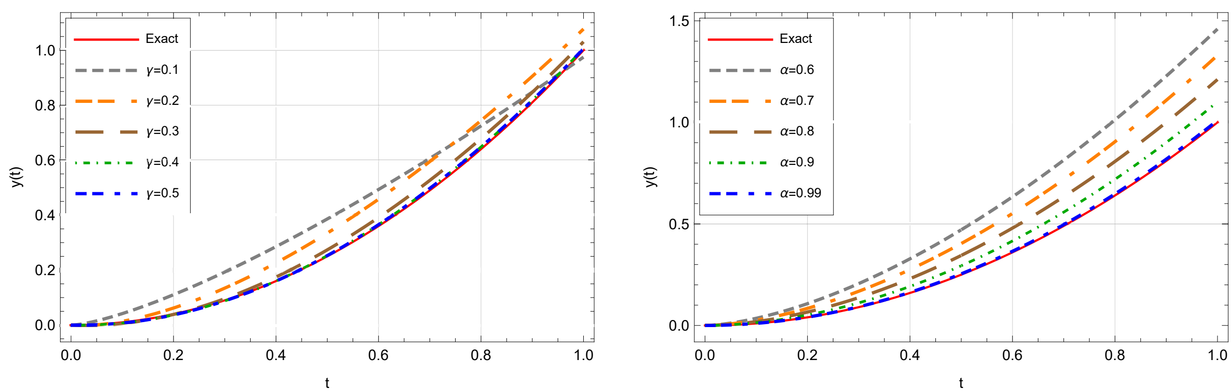

FIGURE 6. (Example 6.3.) Plots of the exact solution together with the approximate solutions obtained by $k = 1$, $M = 2$, $\alpha = 1$ and different values of γ (left) and $k = 2$, $M = 2$, $\gamma = 1$ and different values of α (right).

Example 6.4. Consider Equation (1.3) with [4, 22]:

$$\alpha = \frac{1}{4}, \quad \beta = \frac{1}{2}, \quad f(t,y) = \frac{32}{231\Gamma\left(\frac{3}{4}\right)}(12t+11)t^{7/4} - \frac{8}{105}(6t+7)t^{5/2} - \frac{7}{36}t + \frac{3}{20},$$

$$g(s,y) = \frac{1}{2}y, \quad \kappa(t,s,y) = \frac{1}{3}(t-s)y, \quad T = 1.$$

The exact solution subject to the initial condition $y(0) = 0$ is $y(t) = t^2 + t^3$.

This problem has been solved using the present method with $k = 4$, $M = 2$ ($2^{k-1}(M+1)$ basis functions) and some selected values of γ and the results obtained for the maximum absolute error have been compared with the results of CAS wavelet method ($2^k(2M+1)$ basis functions) in Table 3. Also, with $k = 1$, $M = 12$, a comparison of the obtained results with the results given by Jacobi collocation method [4] ($N+1$ basis functions) is seen in Table 3. From this table, it can be found out our method gives more accurate results comparing to the two other methods with a same number of basis functions. Moreover, the results obtained for $\zeta^{\gamma}_{k,M}$ with $\gamma = \frac{1}{2}$ and different values of k and M are shown in Figure 7.

Example 6.5. Consider Equation (1.3) with [4, 23]

$$0 < \alpha \leq 1, \quad \beta = \frac{1}{2}, \quad f(t,y) = -\frac{2048t^{13/2}}{3003} + 3t^2 - \frac{t}{8},$$

TABLE 3. Comparison of the maximum absolute error for Example 6.4.

Method	Maximum absolute error
CAS wavelet method [22]	
$k=3,\ M=1$	$9.73e{-2}$
Present method	
$k=4,\ M=2,\ \gamma=\frac{1}{2}$	$5.45e{-4}$
$k=4,\ M=2,\ \gamma=\frac{2}{3}$	$1.67e{-4}$
$k=4,\ M=2,\ \gamma=\frac{5}{6}$	$5.63e{-5}$
$k=4,\ M=2,\ \gamma=1$	$3.24e{-5}$
Jacobi collocation method [4]	
$N=12$	$2.73e{-5}$
Present method	
$k=1,\ M=12,\ \gamma=\frac{1}{2}$	$1.97e{-8}$
$k=1,\ M=12,\ \gamma=\frac{2}{3}$	$3.03e{-7}$
$k=1,\ M=12,\ \gamma=\frac{5}{6}$	$6.78e{-7}$
$k=1,\ M=12,\ \gamma=1$	$6.69e{-6}$

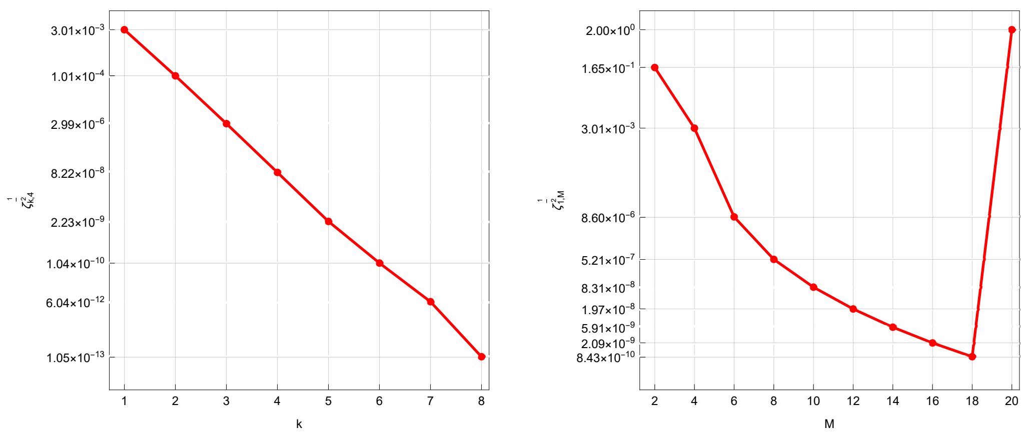

FIGURE 7. (Example 6.4.) Numerical results of the maximum absolute error $\zeta^{\gamma}_{k,M}$ in logarithmic scale with: $M=4$, $\gamma=\frac{1}{2}$ and different values of k (left), $k=1$, $\gamma=\frac{1}{2}$ and different values of M (right).

$$g(s,y)=y^2, \quad \kappa(t,s,y)=tsy^2, \quad T=1,$$

and with the initial condition $y(0)=0$. The exact solution of this problem when $\alpha=1$ is $y(t)=t^3$. For $\gamma=1$, $\alpha=1$, $k=1$ and different values of M, we plotted the absolute error functions with $M=2,3,4,5$ in Figure 8. The method results to the exact solution with $M=6$ (seven basis functions) while the method based on the Jacobi polynomials [4] with the Jacobi parameters equal to zero and $N=11$ (twelve basis functions) leads to a numerical solution with the maximum absolute error equal to 1.07×10^{-11}. Also, to see the effect of increasing the value of k in the accuracy of the method, we plotted the maximum absolute error in a logarithmic scale with $M=4$ and different values of k in Figure 9.

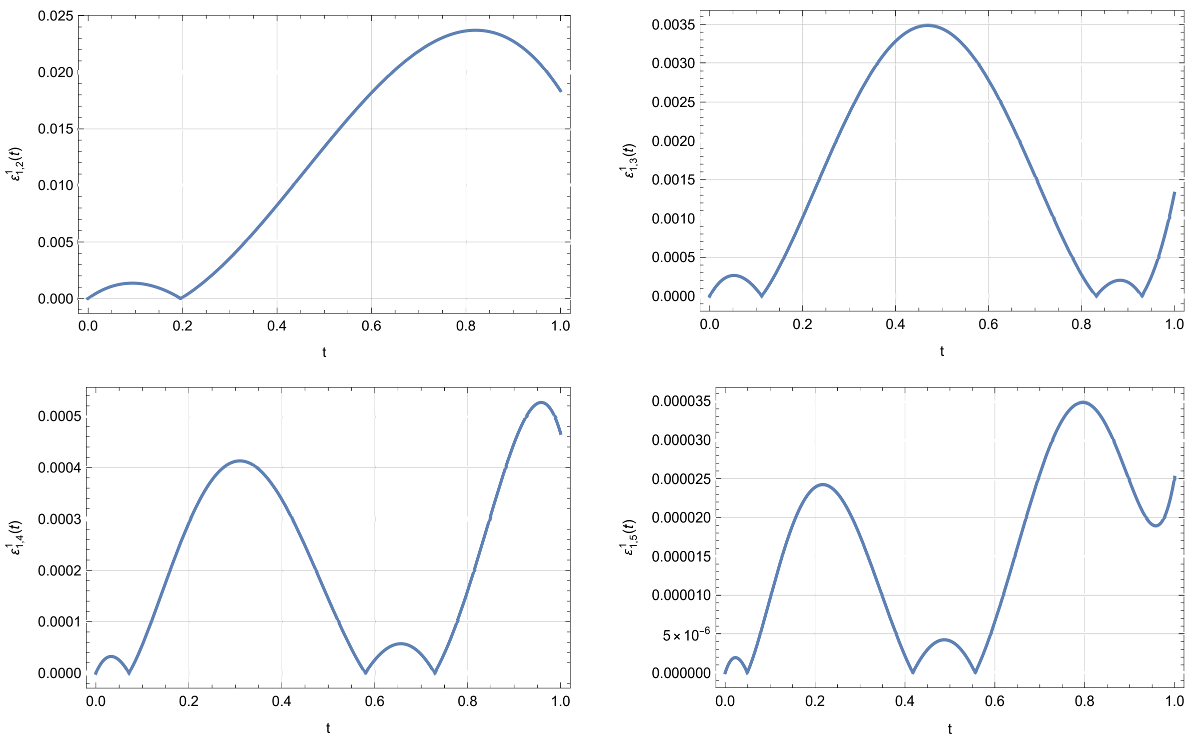

FIGURE 8. (Example 6.5.) Plots of the absolute error function $\varepsilon^{\gamma}_{k,M}(t)$ with $\alpha = 1$, $\gamma = 1$, $k = 1$ and $M = 2, 3, 4, 5$.

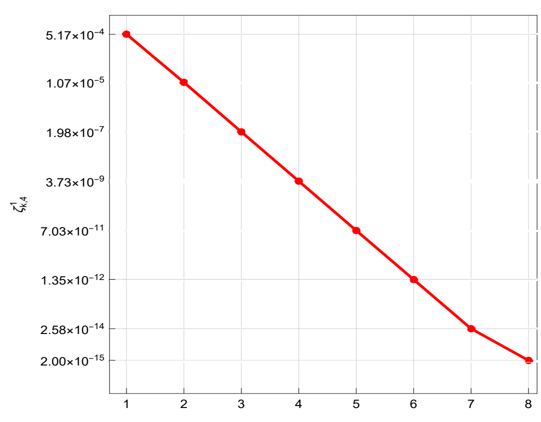

FIGURE 9. (Example 6.5.) Numerical results of the maximum absolute error $\zeta^{\gamma}_{k,M}$ in logarithmic scale with: $M = 4$, $\gamma = 1$ and different values of k.

7. Conclusion

In this work, a new numerical technique based on a generalization of fractional-order Bernoulli wavelets has been proposed for solving two kinds of integro-differential equations. These equations include fractional Volterra weakly singular integro-differential equations of Hammerstein type and fractional Volterra-Fredholm integro-differential equations. The main novelty of the proposed method is to compute the Riemann-Liouville integral of the basis functions without any error that makes this method more accurate than the methods based on the operational matrix of fractional integration. Both problems under study have been easily reduced to systems of algebraic equations using a suitable set of collocation points. An upper error bound has been presented for the approximation of a given function based on our new basis functions. Finally, some examples from the literature have been solved using the new scheme and the obtained results have been reported in some figures and tables. These results show that the method works very well and gives more accurate solutions comparing to the other methods with a same number of basis functions. It has been also noted that with a fixed level of resolution (k) the method will lose the stability with large number of Bernoulli polynomials (M). As a result, taking a small number of Bernoulli polynomials and increasing the level of resolution is a solution to overcome this problem. We insist that the proposed method can be very effective and accurate for solving many practical problems. Therefore, a direction of future research can be applying this method to nonlinear fractional-order partial differential equations appearing in physics and engineering.

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References

[1] D. Baleanu, R. Garra, and I. Petras, *A fractional variational approach to the fractional basset-type equation*, Rep. Math. Phys., *72* (2013), 57–64.

[2] Z. Barary, A. Yazdani Cherati, and S. Nemati, *An efficient numerical scheme for solving a general class of fractional differential equations via fractional-order hybrid Jacobi functions*, Commun. Nonlinear Sci. Numer. Simul., *128* (2024), 107599.

[3] S. Behera and S. Saha Ray, *On a wavelet-based numerical method for linear and nonlinear fractional Volterra integro-differential equations with weakly singular kernels*, Comp. Appl. Math., *41* (2022), 211.

[4] J. Biazar and K. Sadri, *Solution of weakly singular fractional integro-differential equations by using a new operational approach*, J. Comput. Appl. Math., *352* (2019), 453–477.

[5] R. A. Devore and L. R. Scott, *Error bounds for Gaussian quadrature and weighted L^1 polynomial approximation*, SIAM J. Numer. Anal., *21* (1984), 400–412.

[6] R. Hilfer (Ed.), *Applications of fractional calculus in physics*, World Scientific, Singapore, 2000.

[7] P. K. Kythe and P. Puri, *Computational method for linear integral equations*, Birkhäuser, Boston, 2002.

[8] C. Li and M. Cai, *Theory and numerical approximations of fractional integrals and derivatives*, SIAM, 2019.

[9] S. Momani, *Local and global existence theorems on fractional integro-differential equations*, J. Fract. Calc., *18* (2000), 81–86.

[10] S. Momani, A. Jameel, and S. Al-Azawi, *Local and global uniqueness theorems on fractional integro-differential equations via Biharis and Gronwalls inequalities*, Soochow J. Math., *33* (2007), 619–627.

[11] S. Nemati and P. M. Lima, *Numerical solution of nonlinear fractional integro-differential equations with weakly singular kernels via a modification of hat functions*, Appl. Math. Comput., *327* (2018), 79–92.

[12] S. Nemati, S. Sedaghat, and I. Mohammadi, *A fast numerical algorithm based on the second kind Chebyshev polynomials for fractional integro-differential equations with weakly singular kernels*, J. Comput. Appl. Math., *308* (2016), 231–242.

[13] I. Podlubny, *Fractional differential equations, in: Mathematics in science and engineering*, Academic Press, *198* (1999).

[14] O. Postavaru, *Generalized fractional-order hybrid of block-pulse functions and Bernoulli polynomials approach for solving fractional delay differential equations*, Soft Comput., *27* (2023), 737–749.

[15] O. Postavaru and A. Toma, *A numerical approach based on fractional-order hybrid functions of block-pulse and Bernoulli polynomials for numerical solutions of fractional optimal control problems*, Math. Comput. Simul., *194* (2022), 269–284.

[16] P. Rahimkhani, Y. Ordokhani, and E. Babolian, *Fractional-order Bernoulli wavelets and their applications*, Appl. Math. Model., *40* (2016), 8087–8107.

[17] D. Rani and V. Mishra, *Numerical inverse Laplace transform based on Bernoulli polynomials operational matrix for solving nonlinear differential equations*, Results Phys., *16* (2020), 102836.

[18] S. Sabermahani, Y. Ordokhani, and S. A. Yousefi, *Fractional-order Fibonacci-hybrid functions approach for solving fractional delay differential equations*, Eng. Comput., *36* (2020), 795–806.

[19] S. A. Sajjadi, H. Saberi Najafi, and H. Aminikhah, *A numerical study on the non-smooth solutions of the nonlinear weakly singular fractional Volterra integro-differential equations*, Math. Methods Appl. Sci., *46*(4) (2023), 4070–4084.

[20] B. Q. Tang and X. F. Li, *Solution of a class of Volterra integral equations with singular and weakly singular kernels*, Appl. Math. Comput., *199* (2008), 406–413.

[21] Y. Wang and L. Zhu, *SCW method for solving the fractional integro-differential equations with a weakly singular kernel*, Appl. Math. Comput. *275* (2016), 72–80.

[22] M. Yi and J. Huang, *CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel*, Int. J. Comput. Math., *92* (2015), 1715–1728.

[23] M. Yi, L. Wang, and J. Huang, *Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel*, Appl. Math. Model., *40* (2016), 3422–3437.

[24] J. Zhao, J. Xiao, and N. J. Ford, *Collocation methods for fractional integro-differential equations with weakly singular kernels*, Numer. Algorithms, *65* (2014), 723–743.

[25] F. Zhou and X. Xu, *Fractional-order hybrid functions combining simulated annealing algorithm for solving fractional pantograph differential equations*, J. Comput. Sci., *74* (2023), 102172.

[26] V. V. Zozulya and P. I. Gonzalez-Chi, *Weakly singular, singular and hypersingular integrals in 3-d elasticity and fracture mechanics*, J. Chin. Inst. Eng., *22* (1999), 763–775.