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A novel Bernstein operational matrix: applications for conformable fractional calculus

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Abstract

In this work, we focus on the conformable fractional integral and derivative. We approximate the one and two-variable functions by the Bernstein basis and its dual basis while studying convergence. Then, we get the new operational matrix for conformable fractional integral based on the Bernstein basis. To show the effectiveness of these approximations and conformable integral operational matrix, we apply them for solving the nonlinear system of differential equations, the optimal control problem in the conformable fractional sense, and the space conformable fractional telegraph equation.

Keywords. Conformable fractional integral and derivative, Bernstein operational matrix, Conformable fractional differential equation, Fractional conformable optimal control problem, Space conformable fractional telegraph equation.
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1. Introduction

In the late 17th century, the concept of fractional derivatives has proposed. The Riemann-Liouville, Caputo and Grunwald-Letnikov definitions are the common definitions of fractional derivative that are defined as follows, respectively

(1) Riemann Liouville definition:

$$ {}_a^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-x)^{n-\alpha-1} f(x)dx, \quad t>a,\ n-1<\alpha\leq n,\ (n\in N). $$

(2) Caputo definition:

$$ {}_a^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} f(x)dx, \quad t>a,\ n-1<\alpha\leq n,\ (n\in N). $$

(3) Grunwald-Letnikov definition:

$$ {}_a^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha}{j} f(t-jh). $$

However, the loss of some important algebraic features of fractional order differentiation, such as the product rule and the chain rule, is the most important weakness of the Riemann-Liouville fractional derivative and the Caputo fractional derivative. For these important reasons, the authors in [1, 16] introduced a new fractional derivative which is called "the conformable fractional derivative", based on the elementary definition of derivative.

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The study of conformable fractional calculus has led to interesting applications in different fields of science. For example, in 2016, for converting fractional coupled nonlinear Schrodinger equations into the ordinary differential equations, the new conformable fractional derivative was proposed by Eslami in [12]. In 2017, for solving the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities, the modified Kudryashov method was suggested by Hosseini et al. [14]. In a nonlinear differential equation, the existence of a solution for a local fractional nonlinear differential equation with initial condition was proposed by Bayour et al. [9]. Then, Unal et al. [25] obtained an operator method for local fractional linear differential equations. In nonlinear partial differential equations, Cenesiz et al. studied new exact solutions of Burgers-type equations with conformable derivative [10]. Zhao and Luo introduced the general conformable fractional derivative to describe the physical world [27].

In recent years, many problems have been solved by using spectral and pseudo-spectral methods. For instance, in the year 2017, Tabrizidooz et al. applied the pseudo-spectral method for optimal control problems [23, 24]. Pourbabaee et al. proposed a novel Legendre operational matrix for distributed order fractional differential equations [19]. Moreover, a new operational matrix of Riemann-Liouville fractional derivative of orthonormal Bernoulli polynomials has been obtained by them for the numerical solution of some distributed-order time-fractional partial differential equations [20]. Akbari et al. have studied Optimal control and stability analysis of a fractional order mathematical model for infectious disease transmission dynamics [3]. In our last researches, we proposed the new operational matrices based on Caputo derivative, Riemann-Liouville fractional integral and product by the Bernstein basis and applied them for solving fractional quadratic Riccati differential equations [8], multi-order fractional differential equations [21, 22], nonlinear system of fractional differential equations [5] and multi-dimensional fractional optimal control problems with inequality constraint [4, 7]. Now, obtaining some new results for the conformable fractional calculus by the Bernstein operational matrices is the original aim in the present work.

The remainder of this paper are structured as follows. In section 2, we propose several basic definitions and properties of conformable fractional calculus. In section 3, we approximate the functions by Bernstein polynomials with some theorems and corollaries. The new operational matrix for conformable fractional integral based on the Bernstein basis is obtained in section 4. In section 5, we use the obtained outcomes in former sections for solving the nonlinear system of differential equations, the optimal control problem in the conformable fractional sense, and the space conformable fractional telegraph equation. To illustrate the simplicity and precision of the proposed method for solving the different kinds of problems, we perform the numerical simulations by several examples. We propose the conclusion in the final section.

2. Elementary properties and definitions

Some basic definitions and properties of the conformable fractional calculus are remarked in this section. For details, refer to [1, 16].

Definition 2.1. Suppose $f : [a, +\infty) \to R$, then v^{th} order of conformable fractional derivative for f is defined by

$$ {}_aT_t^v f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-v}) - f(t)}{\epsilon}, \quad v \in (0,1]\,,\ t > a, \tag{2.1} $$

and ${}_aT_t^v f(a) = \lim_{t \to a^+} {}_aT_t^v f(t)$.

Definition 2.2. The conformable fractional integral operator of order $v \in (n-1, n]\,, n \in N$ for function $f(t)$, is defined as

$$ {}_aI_t^v f(t) = \frac{1}{(n-1)!} \int_a^t (t-x)^{n-1} (x-a)^{v-n} f(x) dx, \quad t > a, \tag{2.2} $$
$$ {}_aI_t^0 f(t) = f(t), $$

and the conformable fractional derivative of is defined as

$$ {}_aT_t^v f(t) = (t-a)^{n-v} f^{(n)}(t). \tag{2.3} $$

Theorem 2.3. *Let f, g are v-conformable differentiable for $v \in (0,1]$ and $t > 0$. Then*

(1) ${}_aT_t^v(cf+dg)(t) = c\,{}_aT_t^v f(t) + d\,{}_aT_t^v g(t)$ *for all* $c, d \in R$.

(2) ${}_aT_t^v(t-a)^p = p(t-a)^{p-v}$ *for all* $p \in R$.

(3) ${}_aT_t^v(\lambda) = 0$ *for all* $\lambda \in R$.

(4) ${}_aT_t^v(f.g)(t) = f(t)\,{}_aT_t^v g(t) + g(t)\,{}_aT_t^v f(t)$.

(5) ${}_aT_t^v(\frac{f}{g})(t) = \frac{g(t)\,{}_aT_t^v f(t) - f(t)\,{}_aT_t^v g(t)}{g(t)^2}$.

(6) ${}_aT_t^v(f \circ g)(t) = {}_aT_t^v g(t) f'(g(t))$.

Now, we can propose the following properties for $v \in (n-1, n]\,, n \in N$

(1) ${}_aI_t^v(t-a)^\gamma = \frac{\Gamma(v+\gamma-n+1)}{\Gamma(v+\gamma+1)}(t-a)^{v+\gamma}$.

(2) $({}_aT_t^v)({}_aI_t^v)f(t) = f(t)$.

(3) $({}_aI_t^v)({}_aT_t^v)f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a)\frac{(t-a)^k}{k!}, \quad t > a$.

3. Bernstein basis and approximations

Definition 3.1. The m^{th} degree of the Bernstein polynomials (BPs) on $[0,1]$ is defined as:

$$\zeta_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}, \qquad i = 0, 1, \dots, m. \tag{3.1}$$

Corollary 3.2. *Any polynomial* $P(t)$ *of the most degree* m *can be denoted in terms of the basis* $\{\zeta_{0,m}(t), \zeta_{1,m}(t), \dots, \zeta_{m,m}(t)\}$ *as:*

$$P(t) = \sum_{i=0}^{m} c_i \zeta_{i,m}(t). \tag{3.2}$$

Corollary 3.3. *From Eq. (3.1) we have* $\sum_{i=0}^m \zeta_{i,m}(t) = 1$ *and* $\zeta_{i,m}(t) = \sum_{j=1}^m (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i}t^j$.

Lemma 3.4. *(see [17]) For any* $y \in L^2[0,1] = \{y | \int_0^1 y^2(t)dt < \infty\}$ *we have*

$$y(t) \approx \sum_{i=0}^{m} \eta_i \zeta_{i,m} = \eta^T \Psi_m(t), \tag{3.3}$$

where $\Psi_m(t) = [\zeta_{0,m}, \zeta_{1,m}, \dots, \zeta_{m,m}]^T$ *and* $\eta = [\eta_0, \eta_1, \dots, \eta_m]^T$ *is the unique vector and* $\eta^T\Psi_m(t)$ *is the best approximation for* y *out of* $S_m = Span\{\zeta_{0,m}, \zeta_{1,m}, \dots, \zeta_{m,m}\}$.

Noting the fact that Bernstein polynomials are not orthogonal turns out to be their disadvantage when used in the least-squares approximation. As said in [13] one approach to direct least-squares approximation by polynomials in Bernstein form relies on construction of the basis $\{d_{0,m}(x). \dots, d_{m,m}(x)\}$ *that is "dual" to the Bernstein basis of degree* m *on* $[0,1]$.

Corollary 3.5. *In Lemma 3.4, we can get* η_i *as follows*

$$\eta_i = \int_0^1 y(t) d_{i,m}(t)dt, \qquad i = 0, \dots, m, \tag{3.4}$$

where $\{d_{0,m}(x). \dots, d_{m,m}(x)\}$ *has the property*

$$\int_0^1 \zeta_{i,m} d_{j,m}(t)dt = \delta_{ij}, \qquad i, j = 0, \dots, m. \tag{3.5}$$

Moreover, in [15], we can see

$$d_{j,m}(t)=\sum_{j=0}^{m}\mu_{i,j}\zeta_{j,m}(t), \qquad j=0,\dots,m, \tag{3.6}$$

where

$$\mu_{i,j}=\frac{(-1)^{i+j}}{\binom{m}{i}\binom{m}{j}}\sum_{r=0}^{\min(i,j)}(2r+1)\binom{m+r+1}{m-i}\binom{m-r}{m-i}\binom{m+r+1}{m-j}\binom{m-r}{m-j}, \quad i,j=0\dots,m. \tag{3.7}$$

Lemma 3.6. *(See [6]) Let $y:[0,1]\to R$ and $y\in C^{m+1}[0,1]$. If $\eta^T\Psi_m(t)$ be the best approximation y out of S_m then*

$$\left\|y-\eta^T\Psi_m\right\|_2\leq\frac{\hat{K}}{(m+1)!\sqrt{2m+3}}, \tag{3.8}$$

where $\hat{K}=\max_{x\in[0,1]}\left|y^{(m+1)}(x)\right|$. It is clear that $\left\|y-\eta^T\Psi_m\right\|\to 0$ as $m\to\infty$.

Corollary 3.7. *Let $f(t,x)\in L^2[0,1]\times[0,1]$. Then, the unique matrix K exists as:*

$$K=\begin{bmatrix}k_{00} & k_{01} & \dots & k_{0m}\\ k_{10} & k_{11} & \dots & k_{1m}\\ \vdots & \vdots & \vdots & \vdots\\ k_{m0} & k_{m1} & \dots & k_{mm}\end{bmatrix}, \tag{3.9}$$

such that

$$f(t,x)\approx\sum_{i=0}^{m}\xi_{i,m}(t)k_{ij}\xi_{i,m}(x)=\Psi_m(t)^TK\Psi_m(x),$$
$$k_{ij}=\int_0^1\int_0^1 f(t,x)d_{i,m}(t)d_{j,m}(x)dtdx, i,j=0,\dots,m,$$

and $\Psi_m(t)^TK\Psi_m(x)$ is the best approximation f out of $S_m^{t,x}=Span\{(\zeta_{i,m}(t)\zeta_{j,m}(x))_{i,j=0}^m\}$.

Lemma 3.8. *Let $f:[0,1]\times[0,1]\to R$ and $f\in C^{m+1}[0,1]\times[0,1]$. If $\Psi_m(t)^TK\Psi_m(x)$ be the best approximation f out of $S_m^{t,x}$ then*

$$\left\|f(t,x)-\Psi_m(t)^TK\Psi_m(x)\right\|_2\leq\frac{2^{m+1}\bar{M}}{(m+1)!}, \tag{3.10}$$

where $\bar{M}=\max\left\{\left|\frac{\partial^{m+1}f(t,x)}{\partial^\alpha t\partial^\beta x}\right| t,x\in[0,1],\alpha,\beta\in\{0,1,\dots,m+1\},\alpha+\beta=m+1\right\}$. Also, if $f\in C^\infty[0,1]\times[0,1]$ then the error bound vanishes.

Proof. From Taylor expansion of $f(t,x)$ about $(0,0)$, we have

$$f(t,x)=\underbrace{\sum_{i=0}^{m}\frac{1}{i!}\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}\right)^i f(0,0)}_{\tilde{f}}+R_m(t,x), \tag{3.11}$$

where $R_m(t,x)=\frac{1}{(m+1)!}\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}\right)^{m+1}f(\varepsilon_t,\varepsilon_x)$ for some $(\varepsilon_t,\varepsilon_x)\in(0,t)\times(0,x)$. Since $\Psi_m(t)^TK\Psi_m(x)$ is the best approximation f out of $S_m^{t,x}$ and also $\tilde{f}\in S_m^{t,x}$ is as an approximation of f, so we can get

$$\left\|f(t,x)-\Psi_m(t)^TK\Psi_m(x)\right\|_2\leq\|f(t,x)-\tilde{f}(t,x)\|_2$$
$$=\left(\int_0^1\int_0^1|f(t,x)-\tilde{f}(t,x)|^2dtdx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^1 \int_0^1 \left| \frac{1}{(m+1)!} \left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \right)^{m+1} f\left(\varepsilon_t, \varepsilon_x\right) \right|^2 dtdx \right)^{\frac{1}{2}}$$

$$\leq \frac{2^{m+1}}{(m+1)!}\bar{M}. \tag{3.12}$$

Therefore the proof is complete. □

Lemma 3.9. *(See [7]) Let $\hat{A}_{(m+1)\times(m+1)}$ is the product operational matrix with respect to $a_{(m+1)\times 1}$ based on BPs. Then $\hat{A}$ can be obtained as:*

$$a^T \Psi_m(t)\Psi_m(t)^T \approx \Psi_m(t)^T \hat{A}. \tag{3.13}$$

Corollary 3.10. *Let $f(t) \approx f^T \Psi_m(t), g(t) \approx g^T \Phi_m(t)$, then We can approximate the functions $f(t)g(t)$ and $f^k(t)$ as follows:*

$$f(t)g(t) \approx \Psi_m(t)^T \hat{F} g, \tag{3.14}$$

$$f^k(t) \approx \Psi_m(t)^T \hat{F}^{k-1} f.$$

These approximations in this section can be generalized for three-variable functions. Refer to the Appendix for details.

4. Bernstein operational matrix of conformable fractional integral

The aim of this section is to obtain the operational matrix of the conformable fractional integral by BPs. By Corollary 3.3 and Eq. (2.1) we get

$$\begin{aligned} {}_0I_t^v \xi_{i,m}(t) &= \sum_{j=i}^{m} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i} {}_0I_t^v \left(t^j\right) \\ &= \sum_{j=i}^{m} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i} \frac{\Gamma(\alpha+j-n+1)}{\Gamma(\alpha+j+1)} t^{j+v}, \qquad i = 0, \dots, m. \end{aligned} \tag{4.1}$$

Now, we approximate $t^{j+v}(j = 0, \dots, m)$ by BPs as:

$$t^{j+v} \approx P_j^T \Psi_m(t), \tag{4.2}$$

where $P_j = [p_{0,j}, \dots, p_{m,j}]^T, (j = 0, \dots, m)$ and

$$\begin{aligned} p_{l,j} &= \int_0^1 t^{j+v} d_{l,m}(t)dt = \sum_{k=0}^{m} \mu_{l,k} \int_0^1 t^{j+v}\xi_{k,m}(t)dt \\ &= \sum_{k=0}^{m} \mu_{l,k} \sum_{s=k}^{m} (-1)^{s-k} \binom{m}{k} \binom{m-k}{s-k} \int_0^1 t^{j+v+s} dx \\ &= \sum_{k=0}^{m} \mu_{\ell,k} \sum_{s=k}^{m} (-1)^{s-k} \binom{m}{k} \binom{m-k}{s-k} \frac{1}{(j+v+s+1)} \\ &= \sum_{k=0}^{m} \mu_{\ell,k} \psi_{k,j}, \end{aligned} \tag{4.3}$$

and $\Psi_{k,j} = \sum_{s=k}^{m} (-1)^{s-k} \binom{m}{k} \binom{m-k}{s-k} \int_0^1 t^{j+v+s}$.
By Eq. (4.1)–(4.3) we can write

$${}_0I_t^v \xi_{i,m}(t) = \sum_{j=i}^{m} \sum_{l=0}^{m} (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i} \frac{\Gamma(j+1)}{\Gamma(j+v+1)} p_{\ell,j}\xi_{\ell,m}(t),$$

$$= \sum_{\ell=0}^{m} \left(\sum_{j=i}^{m} \Lambda_{i,j,l} \right) \xi_{\ell,m}(t), \text{ for } i = 0, \ldots, m, \tag{4.4}$$

where $\Lambda_{i,j,\ell} = (-1)^{j-i} \binom{m}{i} \binom{m-i}{j-i} \frac{\Gamma(j+1)}{\Gamma(j+v+1)} \sum_{k=0}^{m} \mu_{\ell,k} \psi_{k,j}$.

Finally, from Eq(4.4), we obtain

$$_0I_t^v \Psi_m(t) \approx F_\nu \Psi_m(t), \tag{4.5}$$

where

$$F_\nu = \begin{bmatrix} \sum_{j=0}^{m} \Lambda_{0,j,0} & \sum_{j=0}^{m} \Lambda_{0,j,1} & \cdots & \sum_{j=0}^{m} \Lambda_{0,j,m} \\ \sum_{j=1}^{m} \Lambda_{1,j,0} & \sum_{j=1}^{m} \Lambda_{1,j,1} & \cdots & \sum_{j=1}^{m} \Lambda_{1,j,m} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{j=i}^{m} \Lambda_{i,j,0} & \sum_{j=i}^{m} \Lambda_{i,j,1} & \cdots & \sum_{j=i}^{m} \Lambda_{i,j,m} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{j=m}^{m} \Lambda_{m,j,0} & \sum_{j=m}^{m} \Lambda_{m,j,1} & \cdots & \sum_{j=m}^{m} \Lambda_{m,j,m} \end{bmatrix}. \tag{4.6}$$

5. Applications

Now, we use the obtained operational matrices and approximations in section 4 for solving several problems in engineering with more application and importance in conformable fractional sense, e.g. system of conformable fractional differential equations, conformable fractional optimal control problems and space conformable fractional telegraph equation. We use the absolute error $|u_{exact} - u_{approximate}|$ to show the accuracy of the obtained approximate solutions. It is notable that the used PC is Intel(R) Core(TM) i7-7700K CPU 4.20 GHz. Also we apply version 13 of the Mathematica software for obtaining the results.

5.1. **System of conformable fractional differential equations.** We consider the nonlinear system of conformable fractional differential equations as follows:

$$_0T_t^{v_i} x_i(t) = g_i(t, X(t)), i = 1, \ldots, n, \quad 0 < t \le 1, 0 < v_i \le 1, \tag{5.1}$$

and

$$X(0) = X_0, \tag{5.2}$$

where $X(t) = [x_1(t), \ldots, x_n(t)]^T$ and $X_0 = [x_{0,1}, \ldots, x_{0,n}]^T$. Also, $g_i : [0,1] \times R^n \to R$ is a polynomial. Now, from Eq. (5.2) we define

$$x_i(t) = x_{0,i} + \tilde{x}_i(t), i = 1, 2, \ldots n. \tag{5.3}$$

Substituting Eq. (5.3) in Eqs. (5.1) and (5.2), the problem is reduced as:

$$_0T_t^{v_i} \tilde{x}_i(t) = f_i(t, \tilde{X}(t)), i = 1, \ldots, n, \quad 0 < t \le 1, 0 < v_i \le 1, \tag{5.4}$$

and

$$\tilde{x}_i(0) = 0, i = 1, \ldots, n, \tag{5.5}$$

where $\tilde{X}(t) = [\tilde{x}_1(t), \ldots, \tilde{x}_n(t)]^T$ and $f_i : [0,1] \times R^n \to R$ a polynomial. Now, we use the approximations:

$$_0T_t^{v_i} \tilde{x}_i(t) \approx C_i^T \Psi_m(t), \quad i = 1, \ldots, n, \tag{5.6}$$

where $C_i \in R^{(m+1)\times 1}$. By Eq. (2.3), Eq. (5.6), and Eq. (4.5), we have

$$\tilde{y}_i(t) = {_0I_t^{v_i}} {_0T_t^{v_i}} \tilde{x}_i(t) \approx {_0I_t^{v_i}} \left(C_i^T \Psi_m(t) \right) = C_i^T {_0I_t^{v_i}} \Psi_m(t) \approx C_i^T F_{v_i} \Psi_m(t). \tag{5.7}$$

So, by Eq. (5.6) and Eq. (5.7), the problem Eq. (5.1) and Eq. (5.2) are simplified to:

$$C_i^T \Psi_m(t) = f_i \left(t, C_1^T F_{v_1} \Psi_m(t), \ldots, C_n^T F_{v_n} \Psi_m(t) \right), \qquad i = 1, \ldots, n. \tag{5.8}$$

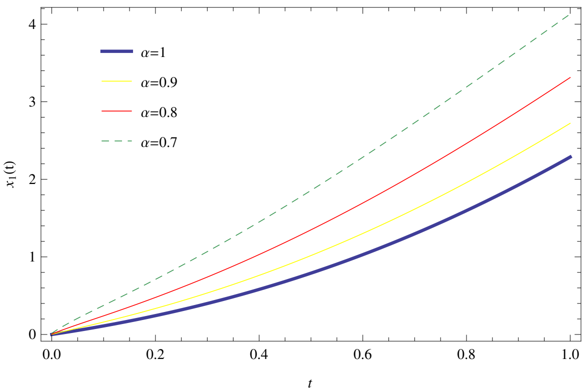

FIGURE 1. Plot of $x_1(t)$ for $m = 10$ and different orders $v_1 = v_2 = \alpha$ in Example 5.1.

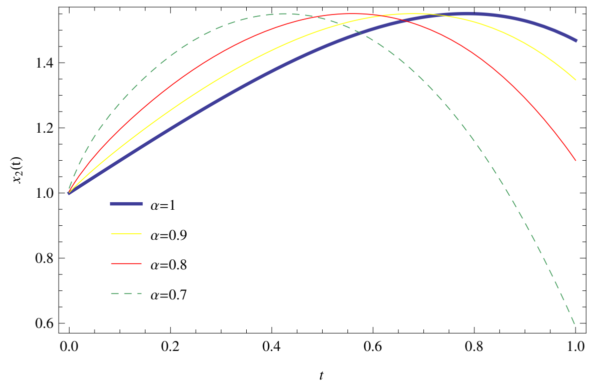

FIGURE 2. Plot of $x_2(t)$ for $m = 10$ and different orders $v_1 = v_2 = \alpha$ in Example 5.1.

Thus, Lemma 3.4 makes the approximations for all of the known functions in Eq. (5.8). Then, by this fact that f_i are polynomial with Corollary 3.10, we conclude the approximations:

$$f_i(t, \tilde{X}(t)) \approx \tilde{F}_i\left(C_1, \dots, C_n\right) \Psi_m(t), i = 1, \dots, n, \tag{5.9}$$

where $\tilde{F}_i : R^{(m+1)\times n} \to R^{1\times(m+1)}$. So, from Eq. (5.8) and Eq. (5.9) we have

$$\left(C_i^T - \tilde{F}_i\left(C_1, \dots, C_n\right)\right) \Psi_m(t) = 0, \quad i = 1, \dots, n. \tag{5.10}$$

Finally, we succeed to simplify the original problem (5.1) and (5.2) to the following algebraic system:

$$C_i^T - \tilde{F}_i\left(C_1, \dots, C_n\right) = 0, \tag{5.11}$$

By solving the system Eq. (5.11) with respect to C_i, we can get the approximate solutions of $y_i(t)$ as:

$$x_i(t) \approx x_{0,i} + C_i^T F_{v_i} \Psi_m(t), \quad i = 1, 2, \dots, n. \tag{5.12}$$

Example 5.1. Consider system of conformable fractional differential equations (for $v_i = 1$ [11], [28])

$$\begin{aligned} {}_0T_t^{v_1} x_1(t) &= x_1(t) + x_2(t), \\ {}_0T_t^{v_2} x_2(t) &= -x_1(t) + x_2(t), \quad 0 < v_1, v_2 \le 1, \end{aligned}$$

and

$$x_1(0) = 0, \quad x_2(0) = 1,$$

with the exact solution for $v_1 = v_2 = 1$ as follows

$$\begin{aligned} x_1(t) &= e^t \sin(t), \\ x_2(t) &= e^t \cos(t). \end{aligned}$$

The obtained solutions for $x_1(t)$ and $x_2(t)$ by present method for $m = 10$ and $v_1 = v_2 = 0.7, 0.8, 0.9, 1$ are depicted in Figures 2 and 3 respectively. In Figures 4 and 5, we display the absolute errors of results for $m = 10$. These figures show that the results have high accuracy and they are good agreement with analytical solutions. As we expected, the approximate solutions near to the exact solutions for $v_1 = v_2 = 1$ as v_1, v_2 approach to 1. To show the validity of the obtained error bound, the absolute error bound and absolute error in norm 2 for $m = 10$ are reported in the Table 1.

Example 5.2. Consider the following system (for $v_i = 1$ [28])

$$\begin{aligned} T_t^{v_1} x_1(t) &= x_1(t), \\ T_t^{v_2} x_2(t) &= 2x_1^2(t), \end{aligned}$$

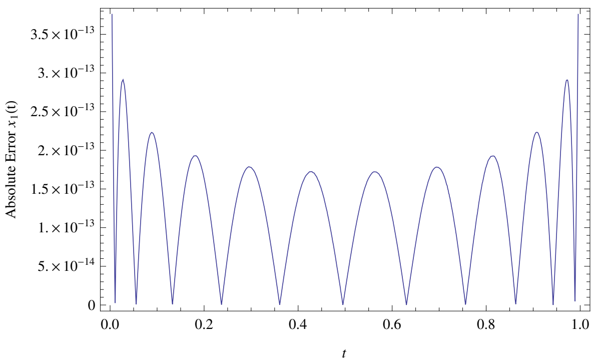

FIGURE 3. Display of absolute error for $x_1(t)$, $v_1 = v_2 = 1$ and $m = 10$ in Example 5.1.

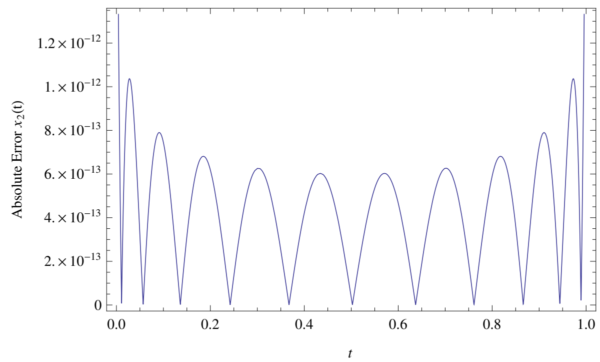

FIGURE 4. Display of absolute error for $x_2(t)$, $v_1 = v_2 = 1$ and $m = 10$ in Example 5.1.

TABLE 1. The absolute error bound and absolute error in norm 2 for m=5, 10 in Example 5.1.

x_i	Absolute error bound m=5	Absolute error m=5	Absolute error bound m=10	Absolute error m=10
x_1	4.77931×10^{-3}	4.78245×10^{-6}	1.67159×10^{-7}	1.50704×10^{-13}
x_2	7.04887×10^{-3}	2.74753×10^{-6}	6.27858×10^{-7}	5.32105×10^{-13}

TABLE 2. The absolute error bound and absolute error in norm 2 for m=10 in Example 5.2.

x_i	Absolute error bound	Absolute error
x_1	1.41996×10^{-8}	1.22762×10^{-14}
x_2	7.90495×10^{-5}	4.21361×10^{-11}
x_3	1.85865×10^{-3}	6.17535×10^{-9}

$$T_t^{T_3} x_3(t) = 3x_1(t)x_2(t), \quad 0 < v_1, v_2, v_3 \leq 1,$$

subject to:

$$x_1(0) = 1, \quad x_2(0) = 1, \quad x_3(0) = 0.$$

For $v_1 = v_2 = v_3 = 1$, the exact solution is

$$x_1(t) = e^t,$$
$$x_2(t) = e^{2t},$$
$$x_3(t) = e^{3t} - 1.$$

Using proposed method the approximations of $x_1(t)x_2(t)$ and $x_3(t)$ for $m = 10$ with different orders of v_1, v_2, and v_3 are plotted in Figures 5–7, respectively. In Figures 8–10, absolute errors of approximations for $m = 10$ and $v_1 = v_2 = v_3 = 1$ depict the exact and the approximate solutions overlap. From these results, we find out that v_1, v_2, and v_3 approach to 1, the approximations close to the exact solutions for $v_1 = v_2 = v_3 = 1$ as expected. To show the validity of the obtained error bound, the absolute error bound and absolute error in norm 2 for $m = 10$ are reported in the Table 2.

5.2. **Conformable fractional optimal control problems (CFOCP).** The purpose of this section is to apply the present scheme for solving the following fractional optimal control problem:

$$\text{Min} \quad \mathrm{M}(x(\tau), u(\tau)) = \frac{1}{2} \int_0^1 \left(x^2(\tau) + u^2(\tau)\right) d\tau, \tag{5.13}$$

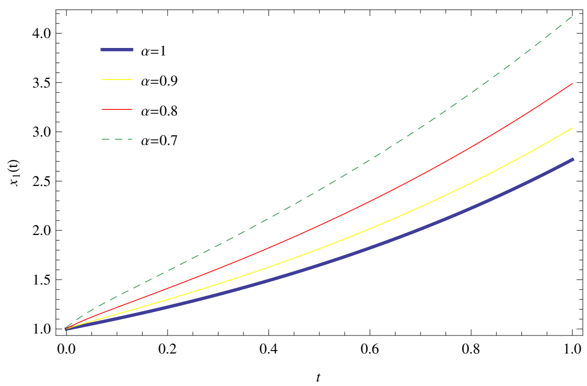

FIGURE 5. Plot of $x_1(t)$ for $m = 10$ and different orders $v_1 = v_2 = v_3 = \alpha$ in Example 5.2.

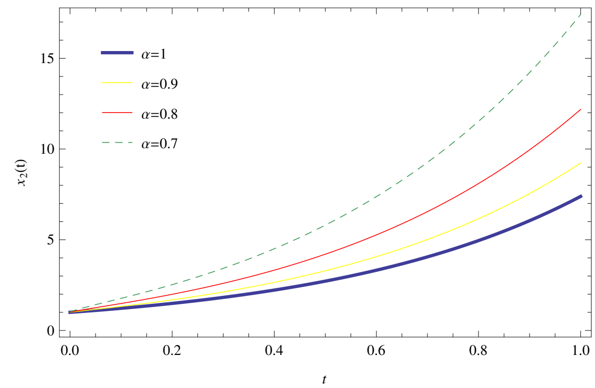

FIGURE 6. Plot of $x_2(t)$ for $m = 10$ and different orders $v_1 = v_2 = v_3 = \alpha$ in Example 5.2.

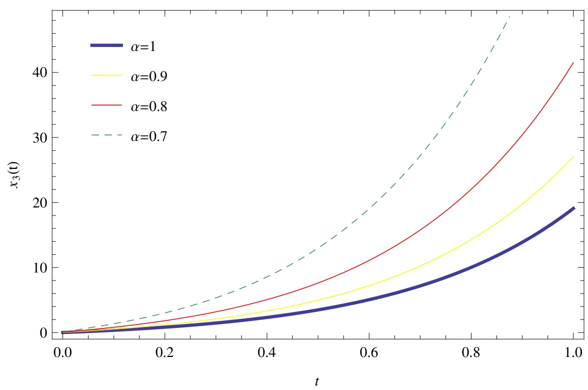

FIGURE 7. Plot of $x_3(t)$ for $m = 10$ and different orders $v_1 = v_2 = v_3 = \alpha$ in Example 5.2.

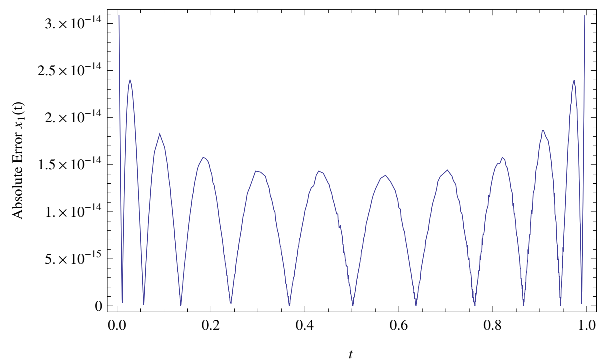

FIGURE 8. Display of absolute error for $x_1(t)$, $v_1 = v_2 = v_3 = 1$ and $m = 10$ in Example 5.2.

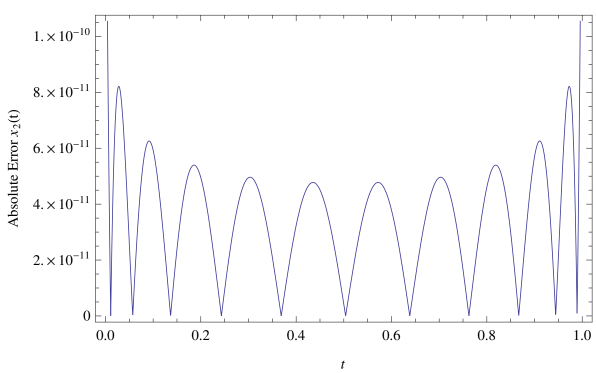

FIGURE 9. Display of absolute error for $x_2(t)$, $v_1 = v_2 = v_3 = 1$ and $m = 10$ in Example 5.2.

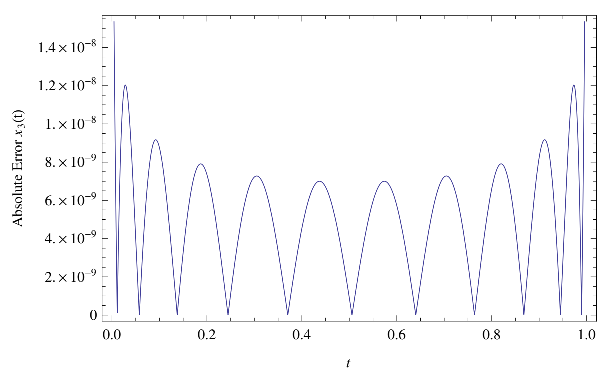

FIGURE 10. Display of absolute error for $x_3(t)$, $v_1 = v_2 = v_3 = 1$ and $m = 10$ in Example 5.2.

subject to

$$ {}_0T_\tau^\alpha x(\tau) = \gamma_1(\tau)x(\tau) + \gamma_2(\tau)u(\tau), \quad \tau, \alpha \in (0,1], \tag{5.14} $$

and

$$ x(0) = x_0, \tag{5.15} $$

where $\gamma_1(\tau)$ and $\gamma_2(\tau)$ are given functions and the unknown functions $x(\tau)$ and $u(\tau)$ are the state and control functions, respectively.

By Eq. (3.3), we can apply the following approximations:

$$ {}_0T_\tau{}^\alpha x(\tau) \approx c^T\Psi_m(\tau), \tag{5.16} $$

$$ u(\tau) \approx b^T\Psi_m(\tau), \tag{5.17} $$

$$ \gamma_1(\tau) \approx \gamma_1^T\Psi_m(\tau), \tag{5.18} $$

$$ \gamma_2(\tau) \approx \gamma_2^T\Psi_m(\tau), \tag{5.19} $$

where $c, b, \gamma_1, \gamma_2 \in R^{(m+1)\times 1}$.

From Eq. (2.3), Eq. (5.16), Corollary 3.3, and Eq. (4.5), we have

$$ \begin{aligned} x(\tau) &= {}_0I_\tau^\alpha T_\tau^\alpha x(\tau) + x_0 \\ &\approx {}_0I_\tau^\alpha \left(c^T\Psi_m(\tau)\right) + x_0\Lambda_m^T\Psi_m(\tau) \\ &= c^T {}_0I_\tau^\alpha\Psi_m(\tau) + x_0\Lambda_m^T\Psi_m(\tau) \approx \underbrace{\left(c^TF_\alpha + x_0\Lambda_m^T\right)}_{c_\alpha^T}\Psi_m(\tau), \end{aligned} \tag{5.20} $$

where $\Lambda_m = \underbrace{[1,1,\ldots,1]^T}_{m+1}$.

Thus, the problem Eq. (5.13)–(5.15) is converted to:

$$ \text{Min}\,\frac{1}{2}\int_0^1 \left(c_\alpha^T\Psi_m(\tau)\Psi_m(\tau)^Tc_\alpha + b^T\Psi_m(\tau)\Psi_m(\tau)^Tb\right)d\tau, \tag{5.21} $$

with constraint

$$ c^T\Psi_m(\tau) = \gamma_1^T\Psi_m(\tau)\Psi_m(\tau)^Tc_\alpha + \gamma_2^T\Psi_m(\tau)\Psi_m(\tau)^Tb. \tag{5.22} $$

So,

$$ \begin{aligned} \text{Min}\,\tilde{\text{M}}(c,b) &= \frac{1}{2}c_\alpha^T\left(\int_0^1 \Psi_m(\tau)\Psi_m(\tau)^Td\tau\right)c_\alpha + \frac{1}{2}b^T\left(\int_0^1 \Psi_m(\tau)\Psi_m(\tau)^Td\tau\right)b \\ &= \frac{1}{2}c_\alpha^T\Theta c_\alpha + \frac{1}{2}b^T\Theta b. \end{aligned} \tag{5.23} $$

where $\Theta \in R^{(m+1)\times(m+1)}$ is called dual matrix based on the Bernstein basis and

$$ \Theta_{i+1,j+1} = \int_0^1 \xi_{i,m}(\tau)\xi_{j,m}(\tau)d\tau = \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}, \quad i,j = 0,1,\cdots m. $$

Moreover, applying Corollary 3.10 in Eq. (5.22) we get

$$ c^T\Psi_m(\tau) = \Psi_m(\tau)^T\hat{\gamma}_1c_\alpha + \Psi_m(\tau)^T\hat{\gamma}_2b. \tag{5.24} $$

Now, we can get the following system from Eq. (5.24) as:

$$ c - \hat{\gamma}_1c_\alpha - \hat{\gamma}_2b = 0. \tag{5.25} $$

Thus, the problem Eqs. (5.13)–(5.15) is converted to the following optimization:

$$ Min\tilde{\text{M}}(c,b) = \frac{1}{2}c_\alpha^T\Theta c_\alpha + \frac{1}{2}b^T\Theta b, \tag{5.26} $$

TABLE 3. The absolute error bound and absolute error in norm 2 for m=5, 10 in Example 5.3.

x,u	Absolute error bound m=5	Absolute error m=5	Absolute error bound m=10	Absolute error m=10
x	3.08167×10^{-3}	1.72923×10^{-6}	2.31652×10^{-7}	7.72653×10^{-12}
u	1.18896×10^{-3}	5.23094×10^{-7}	1.02666×10^{-7}	3.24623×10^{-12}

subject to

$$c - \hat{\gamma}_1 c_\alpha - \hat{\gamma}_2 b = 0. \tag{5.27}$$

Now, we apply the Lagrange multipliers method for solving Eqs. (5.26) and (5.27). Thus, the Lagrange function is defined as:

$$L(c, b, \lambda) = \frac{1}{2} c_\alpha^T \Theta c_\alpha + \frac{1}{2} b^T \Theta b + \hat{\lambda}^T \left(c - \hat{\gamma}_1 c_\alpha - \hat{\gamma}_2 b\right) = 0. \tag{5.28}$$

From the necessary conditions for the extremum, we have:

$$\frac{\partial L}{\partial c} = 0, \tag{5.29}$$

$$\frac{\partial L}{\partial b} = 0, \tag{5.30}$$

$$\frac{\partial L}{\partial \lambda} = 0. \tag{5.31}$$

Finally, by the system Eqs. (5.29)–(5.31) we obtain c, b and λ. Then from Eqs. (5.20) and (5.17) we can obtain the approximate solutions for $x(\tau)$ and $u(\tau)$, respectively.

Example 5.3. Consider the CFOCP as (in Caputo sense [2, 18])

$$\text{Min M} = \frac{1}{2} \int_0^1 \left(x^2(\tau) + u^2(\tau)\right) d\tau,$$

subject to

$${}_{0t}{}^{\alpha} x(\tau) = -x(\tau) + u(\tau),$$

with

$$x(0) = 1.$$

For $\alpha = 1$, this problem has the exact solution

$$x(\tau) = \cosh(\sqrt{2}\tau) + \omega \sinh(\sqrt{2}\tau),$$

$$u(\tau) = (1 + \sqrt{2}\omega) \cosh(\sqrt{2}\tau) + (\sqrt{2} + \omega) \sinh(\sqrt{2}\tau),$$

where $\omega = -\frac{\cosh(\sqrt{2}) + \sqrt{2} \sinh(\sqrt{2})}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})}$.

Figures 11 and 12 display the approximate solutions for $x(\tau)$, $u(\tau)$ with $m = 10$ and different orders of α, respectively. The absolute error function for $m = 10$ and $\alpha = 1$ are plotted in Figures 13 and 14. From these results, we find out that as $\alpha \to 1$, the approximations approach to the exact solutions for $\alpha = 1$ as expected. To show the validity of the obtained error bound, the absolute error bound and absolute error in norm 2 for $m = 10$ are reported in Table 3. Moreover, in Table 4, we reported the estimated target values of M for $m = 10$.

Example 5.4. In this example we consider the CFOCP (in Caputo sense [2], [18])

$$\text{Min} \quad \text{M} = \frac{1}{2} \int_0^1 \left(x^2(\tau) + u^2(\tau)\right) d\tau,$$

subject to

$${}_{0T}{}^{\alpha} x(\tau) = \tau x(\tau) + u(\tau),$$

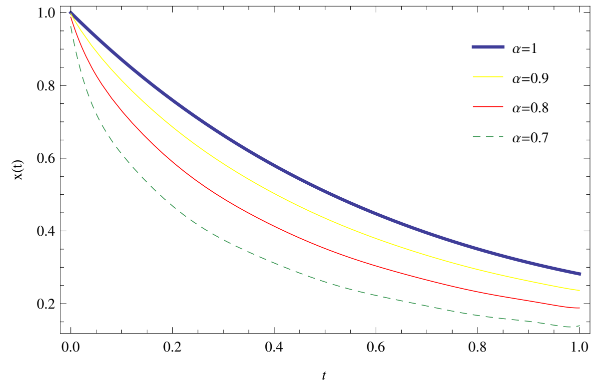

FIGURE 11. Plot of $x(t)$ for $m = 10$ and different orders α in Example 5.3.

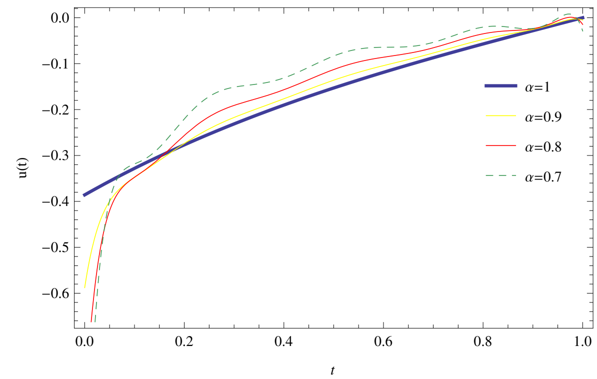

FIGURE 12. Plot of $u(t)$ for $m = 10$ and different orders α in Example 5.3.

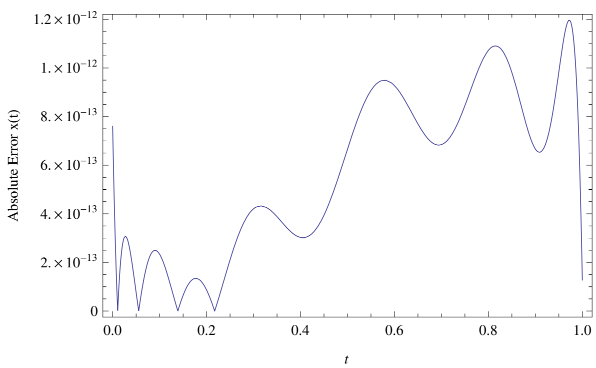

FIGURE 13. Display of absolute error for $x(t)$, $\alpha = 1$ and $m = 10$ in Example 5.3.

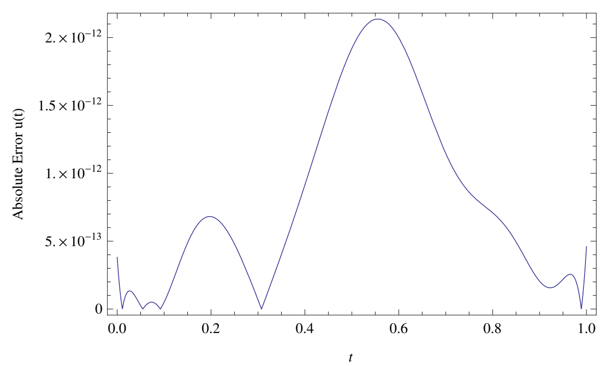

FIGURE 14. Display of absolute error for $u(t)$, $\alpha = 1$ and $m = 10$ in Example 5.3.

TABLE 4. Estimated target values of M for m=5, 10 in Example 5.3.

α	0.7	0.8	0.9	1
M for m=5	0.09357636173418982	0.12855193621636385	0.1621058117640739	0.1929092980927092
M for m=10	0.09262399096041918	0.12828787160106003	0.16206999588757753	0.19290929809289678
Exact	-	-	-	0.19290929809316937

with $x(0) = 1$.

We depict the approximations of $x(t)$ and $u(t)$ by present method in Figures 15 and 16, respectively for $\alpha = 0.7,\ 0.8,\ 0.9,\ 1$. As we have seen in the last examples, we find out that as $\alpha \to 1$, the approximations approach to the exact solutions for $\alpha = 1$. Moreover, in Table 5, we reported the estimated target values of M for $m = 10$.

5.3. **Space conformable fractional telegraph equation.** Now we use the operational matrices method for solving the space conformable fractional telegraph as follows

$$\frac{\partial^2 u(t,x)}{\partial t^2} + p\frac{\partial u(t,x)}{\partial t} + q^2 u(t,x) = {}_0T_x^{2\alpha} u(t,x) + f(t,x), \tag{5.32}$$

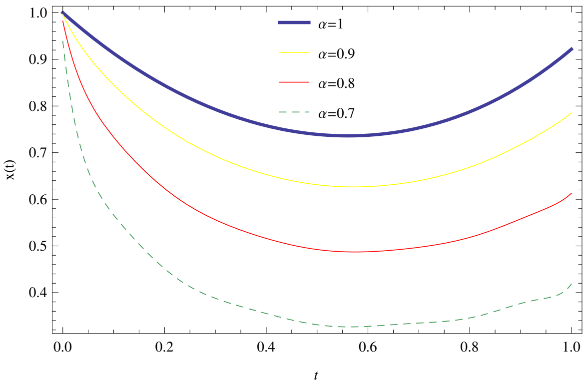

FIGURE 15. Plot of $x(t)$ for $m = 10$ and different orders α in Example 5.4.

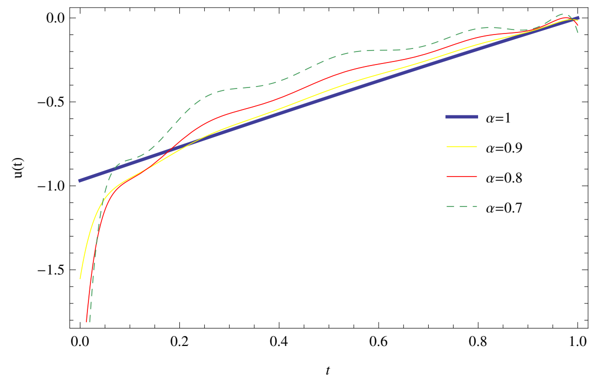

FIGURE 16. Plot of $u(t)$ for $m = 10$ and different orders α in Example 5.4.

TABLE 5. Estimated target values of M for m=5, 10 in Example 5.4.

α	0.7	0.8	0.9	1
M for m=5	0.25696525870031905	0.3516528108742918	0.428573636903991	0.4842676963077697
M for m=10	0.2510608732700348	0.3499612072482705	0.4283557216404006	0.4842676962267954

with conditions

$$u(t,0) = f_1(t), \tag{5.33}$$
$$u_x(t,0) = f_2(t), \tag{5.34}$$

where p, q are constants and also $0.5 < \alpha \leq 1, x, t \in [0,1]$.

Now we can approximate ${}_0T_x^{\alpha}u(t,x)$ by Corollary 3.7 as follows:

$${}_0T_x^{2\alpha}u(t,x) \approx \Psi_m(t)^T K \Psi_m(x), \tag{5.35}$$

where K is unknown matrix. Applying conformable fractional integration of order α with respect to x on Eq. (5.35), we get:

$${}_0I_x^{2\alpha}\,{}_0T_x^{2\alpha}u(t,x) \approx \Psi_m(t)^T K F_{2\alpha}\Psi_m(x), \tag{5.36}$$

so, we have

$$u(t,x) - (C_1 + xC_2) \approx \Psi_m(t)^T K F_{2\alpha}\Psi_m(x). \tag{5.37}$$

The conditions $u(t,0) = f_1(t), u_x(t,0) = f_2(t)$ yield $C_1 = f_1(t), C_2 = f_2(t)$ and Eq. (5.37) can be reduced as

$$u(t,x) \approx \Psi_m(t)^T K F_{\alpha}\Psi_m(x) + f_1(t) + xf_2(t). \tag{5.38}$$

From Corollary 3.7 we approximate $f_1(t) + xf_2(t,x)$ and $f(t,x)$ as following

$$f_1(t) + xf_2(t) \approx \Psi_m(t)^T G_1 \Psi_m(x), \tag{5.39}$$
$$f(t,x) = \Psi_m(t)^T G_2 \Psi_m(x), \tag{5.40}$$

where G_1, G_2 are known matrices. By using Eqs. (5.38)–(5.40) we can write

$$u(t,x) \approx \Psi_m(t)^T \left(KF_{2\alpha} + G_1\right)\Psi_m(x). \tag{5.41}$$

TABLE 6. The absolute error bound and absolute error in norm 2 for m=8 in Example 5.5.

α	Absolute error bound	Absolute error
0.8	1.69312×10^{-3}	9.03679×10^{-5}
0.9	1.69312×10^{-3}	1.45858×10^{-5}
1	1.69312×10^{-3}	1.27606×10^{-15}

In other hand, from section 5 in [26] or section 4 in [7] for $\alpha = 1$ we can apply the following approximations

$$\frac{\partial u(t,x)}{\partial t} \approx \Psi_m(t)^T D^T \left(KF_{2\alpha} + G_1\right) \Psi_m(x), \tag{5.42}$$

and

$$\frac{\partial^2 u(t,x)}{\partial t^2} \approx \Psi_m(t)^T \left(D^2\right)^T \left(KF_{2\alpha} + G_1\right) \Psi_m(x), \tag{5.43}$$

where D is the standard derivative operational matrix based on Bernstein basis, i.e.

$$\frac{d}{dx}\Psi_m(x) \approx D\Psi_m(x). \tag{5.44}$$

Finally, using the approximations Eq. (5.35), and Eqs. (5.40)–(5.43), the problem Eqs. (5.32)–(5.34) reduced as follows

$$\begin{aligned}&\Psi_m(t)^T \left(D^2\right)^T \left(KF_{2\alpha} + G_1\right) \Psi_m(x) + p\Psi_m(t)^T D^T \left(KF_{2\alpha} + G_1\right) \Psi_m(x) + q^2\Psi_m(t)^T \left(KF_{2\alpha} + G_1\right) \Psi_m(x)\\ &= \Psi_m(t)^T K \Psi_m(x) + \Psi_m(t)^T G_2 \Psi_m(x),\end{aligned} \tag{5.45}$$

which can be written as

$$\Psi_m(t)^T \Big(\left(D^2\right)^T \left(KF_{2\alpha} + G_1\right) + pD^T \left(KF_{2\alpha} + G_1\right) + q^2 \left(KF_{2\alpha} + G_1\right) - K - G_2\Big) \Psi_m(x) = 0. \tag{5.46}$$

So, we have

$$\left(D^2\right)^T \left(KF_{2\alpha} + G_1\right) + pD^T \left(KF_{2\alpha} + G_1\right) + q^2 \left(KF_{2\alpha} + G_1\right) - K - G_2 = 0, \tag{5.47}$$

which is an algebraic equation can be easily solved for the unknown matrix K. Applying the value of K in Eq. (5.41) we obtain the approximate solution of the problem Eqs. (5.32)–(5.34).

Example 5.5. Consider the following space conformable fractional telegraph equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial u(t,x)}{\partial t} + u(t,x) = {}_0T_x^{2\alpha} u(t,x) + 2x^3 + 2tx^3 + t^2x^3 - 6t^2x^{3-2\alpha},$$

with conditions

$$u(t,0) = 0,$$
$$u_x(t,0) = 0.$$

where $0.5 < \alpha \leq 1$ and $t, x \in [0,1]$. This problem has the exact solution $u(t,x) = x^3t2$. In Figure 17, we can observe from the contour of absolute error for $\alpha = 1$ and $m = 8$ that the error is less than 10^{-15}. Also we apply the present method to obtain the approximate solutions for the fractional values $\alpha = 0.8, 0.9$ and $m = 8$. The plots of contour of absolute errors for approximate solutions are shown in Figures 18 and 19. The results show that the approximate solutions are in good agreement with the exact solutions for $\alpha = 0.8, 0.9, 1$. To show the validity of the obtained error bound, the absolute error bound and absolute error in norm 2 for $m = 8$ are reported in the Table 6.

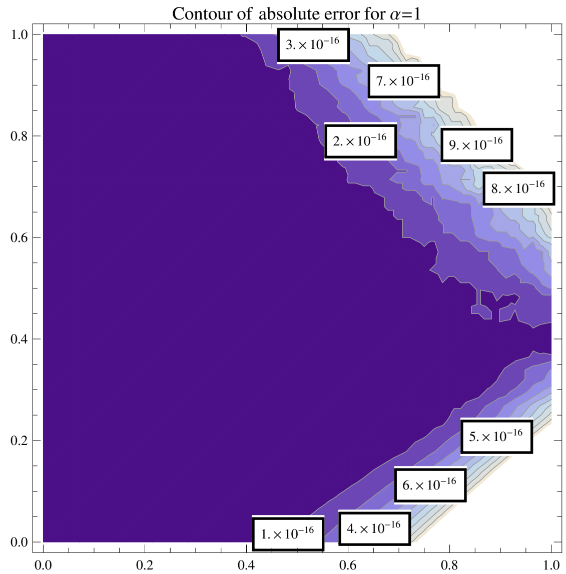

FIGURE 17.

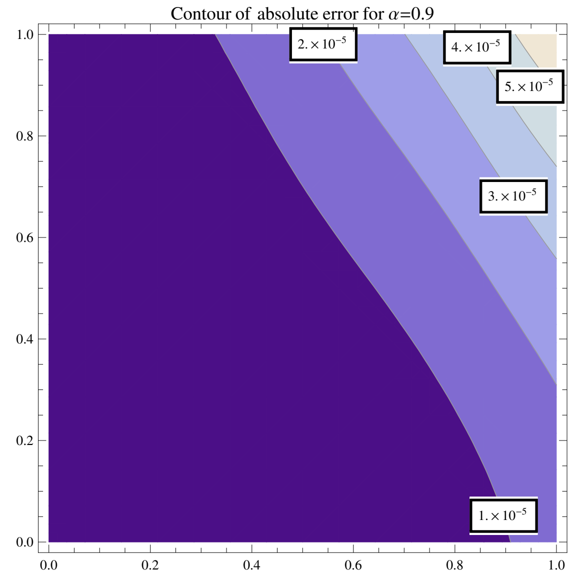

FIGURE 18.

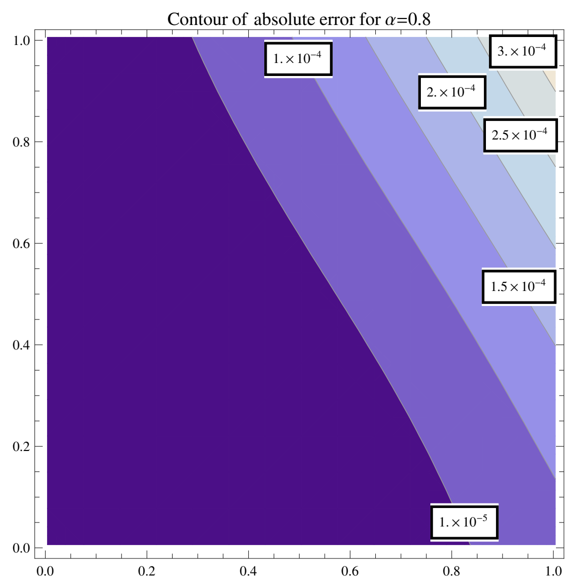

FIGURE 19.

6. Conclusion

In this paper, we proposed the best approximations for one and two-variable functions based on the Bernstein basis. Then, we introduced an algorithm to obtain the conformable fractional integral operational matrix by the Bernstein polynomials. To show the efficiency of the proposed method in implementation, we used the obtained results for solving some interesting problems in engineering, for example, the nonlinear system of conformable fractional differential equations, the conformable fractional optimal control problem, and the space conformable fractional telegraph equation. The results show that the proposed method simply works and is very applicable. Also, we have seen in examples that the approximate solution approaches to the solutions for the integer order of derivate when the order of derivatives approaches to the integer order one, as expected. This method is extensible for solving a wide class of problems in engineering. For example, we can point to problems that dealing with the multi-order conformable fractional derivative or the conformable fractional partial differential equations in higher dimensions.

Conflict of Interest:

I do not have any conflicts of interest.

Appendix

The approximations can be extended to the three-dimensional space and the three-dimensional Bernstein polynomials of order $\mathrm{M} = \mathrm{m} + 1$ are defined as a product function of three Bernstein polynomials

$$\xi_{i,ik}(t,x,y) = \xi_{im}(t)\xi_{i,m}(x)\xi_{kmm}(y), \quad i,j,k = 0,1,2,\ldots,m. \tag{1}$$

The orthogonality condition of $\xi_{i,i,k}(t,x,y)$ is

$$\int_0^1\int_0^1\int_0^1 \xi_{im}(t)\xi_{jm}(x)\xi_{km}(y)d_{pm}(t)d_{q,m}(x)d_{rjm}(y)dtdxdy = \begin{cases} 1, & \text{if } i=p, j=q, k=r, \\ 0, & \text{O.w.} \end{cases} \tag{2}$$

Any $f(t,x,y) \in C\left([0,1]^3\right)$ can be approximated by the polynomials $\xi_{i,j,k}(t,x,y)$ as follows:

$$f(t,x,y) \approx \sum_{i=0}^{m}\sum_{i=0}^{m}\sum_{k=0}^{m} c_{i,j,k}\xi_{im}(t)\xi_{jm}(x)\xi_{km}(y), \tag{3}$$

where

$$\mathrm{c_{i,j,k}} = \int_0^1\int_0^1\int_0^1 f(t,x,y)d_{pm}(t)d_{q\ \mathrm{m}}(x)d_{vm\ \mathrm{m}}(y)dtdxdy. \tag{4}$$

For simplicity, we use the notation $c_{r,n} = c_{i,j,k}$ where $r = i+1$ and $n = Mj + k + 1$, then rewrite (3) as follows:

$$f(t,x,y) \approx \sum_{r=1}^{M}\sum_{n=1}^{M^2} c_{r,n}\xi_{r-1}(t)\xi_n(x,y) = \psi(t)^T K\widehat{\psi}(x,y), \tag{5}$$

where $K_{M\times M^2}$ is the coefficient matrix, $\psi(t) = \left(\xi_{0,m}(t), \xi_{1,m}(t), \ldots, \xi_{m,m}(t)\right)^T$ and

$$\widehat{\psi}(x,y) = \left(\psi_{11}(x,y), \ldots, \psi_{1M}(x,y), \psi_{21}(x,y), \ldots \psi_{2M}(x,y), \ldots \psi_{MM}(x,y)\right)^T, \tag{6}$$

where

$$\psi_{i+1,j+1}(x,y) = \xi_{im}(x)\xi_{im}(y), i,j = 0,1,2,\ldots,m. \tag{7}$$

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