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Existence results for nonlinear elliptic $\overrightarrow{p}(\cdot)-$equations

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Abstract

This paper investigates the existence of distributional solutions for a class of nonlinear elliptic $\overrightarrow{p}(\cdot)-$equations, the analysis focuses on the right-hand side, which comprises of a datum $f \in L^{\overrightarrow{p'}(\cdot)}(\Omega)$ that is independent of u, and a compound nonlinear term involving a given function $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, the solution u and its partial derivatives $\partial_i u$, $i \in \{1, \dots, N\}$, where $L^{\overrightarrow{p}(\cdot)}(\Omega)$ and $L^{\overrightarrow{p'}(\cdot)}(\Omega)$ represent the variable exponents anisotropic Lebesgue spaces.

Keywords. Variable exponent, Nonlinear elliptic equation, Anisotropic Lebesgue-Sobolev space, Distributional solution, Existence, Compound nonlinearity.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open Lipschitz domain (i.e. with Lipschitz boundary $\partial\Omega$).

This paper aims to prove the existence of distributional solution to the anisotropic nonlinear elliptic problems of the form

$$\begin{aligned} &-\sum_{i=1}^{N} \partial_i\big(|\partial_i u|^{p_i(x)-2}\partial_i u\big) = f(x) + \sum_{i=1}^{N}\big(|g| + |u| + |\partial_i u|\big)^{p_i(x)-1}, \quad \text{in } \Omega. \\ &u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where, $f \in L^{\overrightarrow{p'}(\cdot)}(\Omega)$ independent of u, and $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, where $L^{\overrightarrow{p}(\cdot)}(\Omega)$, $L^{\overrightarrow{p'}(\cdot)}(\Omega)$ represent the variable exponents anisotropic Lebesgue spaces defined by

$$L^{\overrightarrow{p}(\cdot)}(\Omega) = \bigcap_{i=1}^{N} L^{p_i(\cdot)}(\Omega), \; L^{\overrightarrow{p'}(\cdot)}(\Omega) = \bigcap_{i=1}^{N} L^{p_i'(\cdot)}(\Omega),$$

where $p_i'(\cdot)$ denotes the Hölder congugate of $p_i(\cdot)$, and $\partial_i u = \frac{\partial u}{\partial x_i}$, $i \in \{1, \dots, N\}$.

This paper is concerned with the study of the existence results of a distributional solutions concerning a class of $\overrightarrow{p}(x)-$Laplacian problem (i.e. variable exponents anisotropic Laplace operator equations) characterized by a compound nonlinearity, it should also be noted here that this type of operators has many uses in applied sciences (see[3, 9, 18]), and it represents a generalization of $p(x)-$Laplacian (For more similar problems, you can see, but not limited to, the papers [11–17]). The right-hand side of our problem is given in terms of $L^{\overrightarrow{p'}(\cdot)}-$data and nonlinearity $\big(|g| + |u| + |\partial_i u|\big)^{p_i(x)-1}$ with $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$, where $L^{\overrightarrow{p}(\cdot)}(\Omega)$ and $L^{\overrightarrow{p'}(\cdot)}(\Omega)$ represent the variable exponents anisotropic Lebesgue spaces.

We began our proof in this work by applying Leray-Schauder's fixed point Theorem of existence (For more about fixed point Theorem, can see [19]) in order to prove the existence of a sequence of suitable approximate solutions

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(u_n). Subsequently we derive a priori estimates for u_n and its partial derivatives. Specifically, we establish the boundedness of u_n in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, and prove almost everywhere convergence in $\overline{\Omega}$ for $\partial_i u_n$, $i \in \{1,\dots,N,\}$, which can be strengthened to strong L^1-convergence. Then, we pass to the limit strongly in the terms $|\partial_i u_n|^{p_i(x)-2}\partial_i u_n$, and $\big(|g_n| + |u_n| + |\partial_i u_n|\big)^{p_i(x)-1}$. Thereby we were able to deducing the convergence of u_n to the solution u of (1.1).

The paper is structured as follows: Section 2 covers fundamental concepts about anisotropic variable exponent Lebesgue-Sobolev spaces and their key properties. The main results and their proofs are presented in section 3.

2. Preliminaries and basic concepts

This section reviews anisotropic Lebesgue-Sobolev spaces with variable exponents and their key properties, as detailed in, for example, [4, 5, 7].

First, define

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}, \qquad 1 < p^- \leq p^+ < \infty\},$$

where, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain,

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad \text{and} \qquad p^- = \min_{x\in\overline{\Omega}} p(x).$$

- Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. Then, $\forall \xi, \xi' \in \mathbb{R}$ and $\forall \varepsilon > 0$ the following inequalities are true :
  - $(*)$ Young's inequality :

$$|\xi\xi'| \leq \varepsilon|\xi|^{p(x)} + c(\varepsilon)|\xi'|^{p'(x)}, \tag{2.1}$$

    where, $p'(\cdot)$ denotes the Hölder conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ in $\overline{\Omega}$).
  - $(*)$ In addition :

$$|\xi + \xi'|^{p(x)} \leq 2^{p^+-1}(|\xi|^{p(x)} + |\xi'|^{p(x)}).$$

  - $(*)$ If $(\xi, \xi') \neq (0,0)$,

$$(|\xi|^{p(x)-2}\xi - |\xi'|^{p(x)-2}\xi')(\xi - \xi') \geq \begin{cases} 2^{2-p^+}|\xi - \xi'|^{p(x)}, & \text{if } p(x) \geq 2, \\ (p^- - 1)\frac{|\xi-\xi'|^2}{(|\xi|+|\xi'|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \tag{2.2}$$

- Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}; \rho_{p(\cdot)}(u) < \infty\},$$

where

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx, \text{ is called the convex modular.}$$

It is a Banach and reflexive space when equipped with the Luxemburg norm given by:

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{s > 0 : \rho_{p(\cdot)}(\frac{u}{s}) \leq 1\right\},$$

- The following Hölder type inequality holds :

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

- Next results(see [4, 5]) we need to use them later. Let (u_n), $u \in L^{p(\cdot)}(\Omega)$, then:

$$\min\left(\rho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right), \tag{2.3}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2.4}$$

- We will now define the main spaces in our paper are anisotropic Sobolev spaces with variable exponents $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1, +\infty)\big),\ i \in \{1, \dots, N\}$, and $\forall x \in \overline{\Omega}$ we set that

$$\overrightarrow{p}(x) = (p_1(x), \dots, p_N(x)), \quad p_+(x) = \max_{1\le i\le N} p_i(x),\ p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}, \quad \overline{p}^{\star}(x) = \frac{N\overline{p}(x)}{N-\overline{p}(x)}, \text{if } \overline{p}(x) < N.$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \Big\{ u \in L^{p_+(\cdot)}(\Omega) \text{ and } \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i \in \{1, \dots, N\} \Big\},$$

equipped with the following norm

$$u \mapsto \|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}. \tag{2.5}$$

The Banach space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ (Our results are based on it) is defined as follow

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega),$$

endowed with the norm (2.5).

• The following important results (see [6, 7]) are needed during the proof steps.
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$.

Lemma 2.1. *If we have $r \in \mathcal{C}_+(\overline{\Omega})$ such that $r(\cdot) < \max(p_+(\cdot),\ \overline{p}^{\star}(\cdot))$ in $\overline{\Omega}$. Then*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact embedding.} \tag{2.6}$$

Lemma 2.2. *If we have the following condition*

$$p_+(\cdot) < \overline{p}^{\star}(\cdot) \text{ in } \overline{\Omega}. \tag{2.7}$$

Then, there exists $c >$ independent of u, such that

$$\|u\|_{p_+(\cdot)} \le c\sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)},\ \ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{2.8}$$

Remark 2.3. If (2.7) holds, then (2.8) implies that

$$u \mapsto \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)} \text{ is an equivalent norm to (2.5).} \tag{2.9}$$

3. Statement of results and proofs

Definition 3.1. u is a distributional solution of the problem (1.1) if and only if $u \in W_0^{1,1}(\Omega)$, and for all $\varphi \in C_c^{\infty}(\Omega)$,

$$\sum_{i=1}^{N}\int_{\Omega} |\partial_i u|^{p_i(x)-2}\partial_i u \partial_i \varphi\, dx = \sum_{i=1}^{N}\int_{\Omega} (|g| + |u| + |\partial_i u|)^{p_i(x)-1}\, \varphi\, dx + \int_{\Omega} f(x)\varphi\, dx.$$

Our main result is that:

Theorem 3.2. *Let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega}),\ i \in \{1, \dots, N\}$ such that $\overline{p} < N$ and (2.7) holds, and assume that $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. Then the problem (1.1) has at least one solutions $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the distributional sense.*

Remark 3.3. Condition (2.7) is adopted in our main Theorem in order to consider the norm (2.9) in all steps of our work.

3.1. **Existence of approximate solutions.** Let (f_n) and (g_n) be a two sequences of bounded functions defined in Ω which (f_n), (g_n) converges to f, g in $L^{\overrightarrow{p}'(\cdot)}(\Omega)$, $L^{\overrightarrow{p}(\cdot)}(\Omega)$, respectively.

Remark 3.4. Since $f_n \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, then from (2.3), we obtain

$$\|f_n\|_{p_i'(\cdot)} \leq 1 + \rho_{p_i'(x)}^{\frac{1}{\overline{p'_i}^{-}(\cdot)}}(f_n) \leq 2 + \rho_{p'_i}^{\frac{1}{\overline{p'}^{-}_{-}}}(f_n) < \infty.$$

Through this, we conclude that

$$f_n \text{ is bounded in } L^{p_i'(\cdot)}(\Omega),\ i = 1, \dots, N. \tag{3.1}$$

Similarly, for g_n in $L^{\overrightarrow{p}(\cdot)}(\Omega)$, we obtain

$$g_n \text{ is bounded in } L^{p_i(\cdot)}(\Omega),\ i = 1, \dots, N. \tag{3.2}$$

Lemma 3.5. *Let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$, $i \in \{1, \dots, N\}$ such that $\overline{p} < N$ and (2.7) holds, and assume that $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, $g \in L^{\overrightarrow{p}(\cdot)}(\Omega)$. Then, there exists at least one solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ in the weak sense to the approximated problems*

$$\begin{aligned} &-\sum_{i=1}^{N} \partial_i\big(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\big) = f_n(x) + \sum_{i=1}^{N} \left(|g_n| + |u_n| + |\partial_i u_n|\right)^{p_i(x)-1}, \ \textit{in } \Omega, \\ &u_n = 0, \ \textit{on } \partial\Omega, \end{aligned} \tag{3.3}$$

in the sense that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i(x)-2}\partial_i u_n \partial_i \varphi \, dx = \sum_{i=1}^{N} \int_{\Omega} \left(|g_n| + |u_n| + |\partial_i u_n|\right)^{p_i(x)-1} \varphi \, dx + \int_{\Omega} f_n(x)\varphi \, dx, \tag{3.4}$$

for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Proof. For $n \geq 1$ fixed in $\mathbb{N}$ and $\forall (v, \xi) \in X \times [0, 1]$ which $X = \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we consider the problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i\left(|\partial_i u|^{p_i(x)-2}\partial_i u\right) = \xi\left(f_n + \sum\limits_{i=1}^{N} \left(|g_n| + |v| + |\partial_i v|\right)^{p_i(x)-1}\right), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Let be the operator: $T : X \times [0, 1] \longrightarrow X$ such that:

$$\forall (v, \xi) \in X \times [0, 1] : \ u = T(v, \xi) \Leftrightarrow$$

$$\forall \varphi \in X : \sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i(x)-2}\partial_i u \partial_i \varphi \, dx = \xi\left(\int_{\Omega} f_n \varphi \, dx + \sum_{i=1}^{N} \int_{\Omega} \left(|g_n| + |v| + |\partial_i v|\right)^{p_i(x)-1} \varphi \, dx\right). \tag{3.6}$$

Since it is easy to verify that,

$$\forall (v, \xi) \in X \times [0, 1] : \left(f_n + \sum_{i=1}^{N} \left(|g_n| + |v| + |\partial_i v|\right)^{p_i(x)-1}\right) \in L^{\overrightarrow{p}'(\cdot)}(\Omega),$$

then the main Theorem on monotone operators (see [1, 2, 10, 20]) guarantees us the existence of a weak solution u to the problem (3.5) in X, and its uniqueness results directly from the uniqueness of the solution to the problem $(= 0)$, which results from the assumption that there are two weak solutions to (3.5) with taking into account the above assumption that f is independent of u.

Now we will estimate the solution u. choosing $\varphi = u$ in (3.6), and using (2.7), Hölder inequality, and (2.3), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)}\,dx \leq & 2\|f_n\|_{p_i'(\cdot)}\|u\|_{p_i(\cdot)} + 2\|(|g_n|+|v|+|\partial_i v|)^{p_i(x)-1}\|_{p_i'(\cdot)}\|u\|_{p_i(\cdot)}\\ \leq & 2\|f_n\|_{p_i'(\cdot)}\|u\|_{p_i(\cdot)} + c\left(1+\sum_{i=1}^{N}\int_{\Omega}(|g_n|+|v|+|\partial_i v|)^{p_i(x)}\,dx\right)^{\frac{1}{p'^{-}_{-}}}\|u\|_{p_i(\cdot)}\\ \leq & C\|u\|_{\vec{p}(\cdot)} + C'\left(1+\sum_{i=1}^{N}\int_{\Omega}\left(|g_n|^{p_i(x)}+|v|^{p_i(x)}+|\partial_i v|^{p_i(x)}\right)dx\right)\|u\|_{\vec{p}(\cdot)}.\end{aligned} \tag{3.7}$$

From (2.7), Lemma 2.1, and (2.4), it follows that

$$\begin{aligned}\int_{\Omega}|v|^{p_i(x)}\,dx &\leq 1+\|v\|_{p_i(\cdot)}^{p_i^+(x)}\\ &\leq 2+\|v\|_{p_i(\cdot)}^{p_+^+}\\ &\leq 2+c\|v\|_{\vec{p}(\cdot)}^{p_+^+}.\end{aligned} \tag{3.8}$$

By (2.7), (2.8), and (2.4), we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_i v|^{p_i(x)}\,dx &\leq N+\sum_{i=1}^{N}\|\partial_i v\|_{p_i(\cdot)}^{p_i^+(x)}\\ &\leq 2N+\sum_{i=1}^{N}\|\partial_i v\|_{p_i(\cdot)}^{p_+^+}\\ &\leq 2N+\left(\sum_{i=1}^{N}\|\partial_i v\|_{p_i(\cdot)}\right)^{p_+^+}\\ &= 2N+\|v\|_{\vec{p}(\cdot)}^{p_+^+}.\end{aligned} \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we find that

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)}\,dx \leq C\left(1+\|v\|_{\vec{p}(\cdot)}^{p_+^+}\right)\|u\|_{\vec{p}(\cdot)}. \tag{3.10}$$

From another side, by using (2.4), we can obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)}\,dx \geq \sum_{i=1}^{N}\min\{\|\partial_i u\|_{p_i(x)}^{p_i^-},\|\partial_i u\|_{p_i(x)}^{p_i^+}\}.$$

We put for every $i \in \{1,\dots,N\}$,

$$\eta_i = \begin{cases} p_+^+, & \text{if } \|\partial_i u\|_{p_i(\cdot)} < 1,\\ p_-^-, & \text{if } \|\partial_i u\|_{p_i(\cdot)} \geq 1,\end{cases}$$

we get

$$\sum_{i=1}^{N}\min\{\|\partial_i u\|_{p_i(\cdot)}^{p_i^-},\|\partial_i u\|_{p_i(\cdot)}^{p_i^+}\}\geq\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}^{\eta_i}$$
$$\geq\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}^{p_-^-}-\sum_{\{i,\xi_i=p_+^+\}}(\|\partial_i u\|_{p_i(\cdot)}^{p_-^-}-\|\partial_i u\|_{p_i(\cdot)}^{p_+^+})$$
$$\geq\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}^{p_-^-}-\sum_{\{i:\eta_i=p_+^+\}}\|\partial_i u\|_{p_i(\cdot)}^{p_-^-}\geq(\frac{1}{N}\sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)})^{p_-^-}-N.$$

Then, we get

$$\sum_{i=1}^{N}\int_\Omega|\partial_i u|^{p_i(x)}\,dx\geq\left(\frac{1}{N}\|u\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-}-N. \tag{3.11}$$

From (3.10) and (3.11), we conclude

$$\|u\|_{\overrightarrow{p}(\cdot)}^{p_-^-}\leq C'\left(1+\|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)\|u\|_{\overrightarrow{p}(\cdot)}. \tag{3.12}$$

Then, there exists $c>0$ such that

$$\|u\|_{\overrightarrow{p}(\cdot)}\leq c\left(1+\|v\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right)^{\frac{1}{p_-^--1}}. \tag{3.13}$$

• Prove the continuity of the operator T:
Let $n\geq 1$ fixed in $\mathbb{N}$, and let $(v_k,\xi_k)_{k\geq 1}\subset X\times[0,1]$ be a sequence converges to $(v,\xi)\in X\times[0,1]$. Then, we have

$$v_k\longrightarrow v,\quad \text{Strongly}, \tag{3.14}$$
$$\xi_k\longrightarrow \xi,\quad \text{Strongly}. \tag{3.15}$$

We consider the sequence $(u_k)_{\{k\in\mathbb{N}^*\}}$ where $u_k=T(v_k,\xi_k)$. Then, we get $\forall\varphi\in X$;

$$\sum_{i=1}^{N}\int_\Omega|\partial_i u_k|^{p_i(x)-2}\partial_i u_k\partial_i\varphi\,dx=\xi_k\left(\int_\Omega f_n\varphi\,dx+\sum_{i=1}^{N}\int_\Omega(|g_n|+|v_k|+|\partial_i v_k|)^{p_i(x)-1}\varphi\,dx\right). \tag{3.16}$$

By (3.13) and the fact that $\|v_k\|_{\overrightarrow{p}(\cdot)}<+\infty$ (due (3.14)):

$$\|u_k\|_{\overrightarrow{p}(\cdot)}=\|T(v_k,\xi_k)\|_{\overrightarrow{p}(\cdot)}\leq c\left(1+\|v_k\|_{\overrightarrow{p}(x)}^{p_+^+}\right)^{\frac{1}{p_-^--1}}\leq\delta, \tag{3.17}$$

with $\delta>0$ independent of k.

From (3.17) we conclude the boundedness of (u_k) in X.
So, there exists a subsequence (still denoted by (u_k)) and $u\in X$ such that

$$u_k\rightharpoonup u,\quad \text{weakly in } X. \tag{3.18}$$

We now show that

$$\lim_{k\longrightarrow+\infty}\Phi_{i,k}=0, \tag{3.19}$$

where

$$\Phi_{i,k}=\int_\Omega\left(|\partial_i u_k|^{p_i(x)-2}\partial_i u_k-|\partial_i u|^{p_i(x)-2}\partial_i w\right)(\partial_i u_k-\partial_i u)\,dx,\quad i\in\{1,\dots,N,\}.$$

Choosing $\varphi = u_k - u$ in (3.16), yields

$$\sum_{i=1}^{N} \Phi_{i,k} = \xi_k \int_\Omega f_n(u_k - u) + \sum_{i=1}^{N} \int_\Omega (|g_n| + |v_k| + |\partial_i v_k|)^{p_i(x)-1}(u_k - u)\,dx$$
$$+ \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u(\partial_i u_k - \partial_i u)\,dx. \tag{3.20}$$

Since $\|f_n\|_{p_i'(\cdot)} < +\infty$ and $\|(|g_n|+|v_k|+|\partial_i v_k|)^{p_i(x)-1}\|_{p_i'(\cdot)} < +\infty$, $u_k \longrightarrow u$ strongly in $L^{r(\cdot)}(\Omega)$ where $r(\cdot)$ mentioned in Lemma 2.1, the fact that $\||\partial_i u|^{p_i(x)-2}\partial_i u\|_{p_i'(\cdot)} < +\infty$, and (3.18), we conclude that the right side of (3.20) goes to 0 when $k \longrightarrow +\infty$, with this we get (3.19).
Now define

$$\Omega_i^{(1)} = \{x \in \Omega, p_i(x) \geq 2\}, \text{ and } \Omega_i^{(2)} = \{x \in \Omega, 1 < p_i(x) < 2\}.$$

From (2.2), we get

$$2^{2-p_i^+} \int_{\Omega_i^{(1)}} |\partial_i(u_k - u)|^{p_i(x)}\,dx \leq \int_{\Omega_i^{(1)}} \left[|\partial_i u_k|^{p_i(x)-2}\partial_i u_k - |\partial_i u|^{p_i(x)-2}\partial_i u\right]\partial_i(u_k - u)\,dx \leq \Phi_{i,k}. \tag{3.21}$$

On the other hand,

$$\int_{\Omega_i^{(2)}} |\partial_i(u_k - u)|^{p_i(x)}\,dx \leq \int_{\Omega_i^{(2)}} \frac{|\partial_i(u_k - u)|^{p_i(x)}}{(|\partial_i u_k| + |\partial_i u|)^{\frac{p_i(x)(2-p_i(x))}{2}}} (|\partial_i u_k| + |\partial_i u|)^{\frac{p_i(x)(2-p_i(x))}{2}}\,dx$$
$$\leq 2\max\left\{\left(\int_{\Omega_i^{(2)}} \frac{|\partial_i(u_k - u)|^2}{(|\partial_i u_k| + |\partial_i u|)^{2-p_i(x)}}\,dx\right)^{\frac{p_i^-}{2}}, \left(\int_{\Omega_i^{(2)}} \frac{|\partial_i(u_k - u)|^2}{(|\partial_i u_k| + |\partial_i u|)^{2-p_i(x)}}\,dx\right)^{\frac{p_i^+}{2}}\right\}$$
$$\times \max\left\{\left(\int_\Omega (|\partial_i u_k| + |\partial_i u|)^{p_i(x)}\,dx\right)^{\frac{2-p_i^+}{2}}, \left(\int_\Omega (|\partial_i u_k| + |\partial_i u|)^{p_i(x)}\,dx\right)^{\frac{2-p_i^-}{2}}\right\}$$
$$\leq 2c\max\left\{\left(\Phi_{i,k}\right)^{\frac{p_i^-}{2}}, \left(\Phi_{i,k}\right)^{\frac{p_i^+}{2}}\right\} \times \left(1 + (\rho_{p_i}(|\partial_i u_k| + |\partial_i u|))^{\frac{2-p_-}{2}}\right). \tag{3.22}$$

Since u_k, $u \in X$, and (3.19), after letting $k \longrightarrow +\infty$ in (3.21) and in (3.22), we obtain that

$$\lim_{k\to+\infty} \int_\Omega |\partial_i u_k - \partial_i u|^{p_i(x)} = 0, \quad i \in \{1, \ldots, N, \}. \tag{3.23}$$

By using (2.7) and (2.3), we obtain that

$$\|u_k - u\|_{\overrightarrow{p}(\cdot)} = \sum_{i=1}^{N} \|\partial_i(u_k - u)\|_{p_i(\cdot)}$$
$$\leq \sum_{i=1}^{N} \max\left(\rho_{p_i(\cdot)}^{\frac{1}{p_i^+}}(\partial_i u_k - \partial_i u), \rho_{p_i(\cdot)}^{\frac{1}{p_i^-}}(\partial_i u_k - \partial_i u)\right), \tag{3.24}$$

where

$$\rho_{p_i(\cdot)}(\partial_i u_k - \partial_i u) = \int_\Omega |\partial_i(u_k - u)|^{p_i(x)}\,dx, \; i \in \{1, \ldots, N, \}.$$

By combining (3.23) and (3.24), we conclude that

$$\lim_{k\longrightarrow+\infty} \|u_k - u\|_{\overrightarrow{p}(\cdot)} = 0. \tag{3.25}$$

Then, (3.25) implies that

$$u_k \longrightarrow u, \;\; \text{Strongly in } X. \tag{3.26}$$

Since the continuity of the function $w \mapsto \sum_{i=1}^{N}(|g_n|+|w|+|\partial_i w|)^{p_i(x)-1}$ on X, we can pass to the limit in (3.16) when $k \longrightarrow +\infty$, and (3.26), we obtain $\forall \varphi \in X$,

$$\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i(x)-2}\partial_i u \partial_i \varphi \, dx = \xi\left(\int_{\Omega} f_n \varphi \, dx + \sum_{i=1}^{N}\int_{\Omega}(|g_n|+|v|+|\partial_i v|)^{p_i(x)-1}\varphi \, dx\right). \tag{3.27}$$

This means that, $u = T(v,\xi)$.
The uniqueness of the weak solution of (3.5) gives us

$$T(v_k,\xi_k) = u_k \longrightarrow u = T(v,\xi), \quad \text{Strongly in } X. \tag{3.28}$$

Then, (3.28) implies the continuity of T.

• Prove the compactness of the operator T: Let $\widehat{B}$ be a bounded of $X \times [0,1]$. Thus $\widehat{B}$ is contained in a product of the type $B \times [0,1]$ with B a bounded of X, which can be assumed to be a ball of center O and of radius $r > 0$.

For $u \in T(\widehat{B})$, we have, thanks to (3.13):

$$\|u\|_{\overrightarrow{p}(\cdot)} \leq c\left(1 + r^{p_+^+}\right)^{\frac{1}{p_-^- - 1}} = \rho.$$

For $u = T(v,\xi)$ with $(v,\xi) \in B \times [0,1]$ ($\|v\|_{\overrightarrow{p}(\cdot)} \leq r$).

This proves that T applies $\widehat{B}$ in the closed ball of center O and radius ρ (ρ depend on r) in X. Let $(u_k) \subset T(\widehat{B})$ be a sequence, so $u_k = T(v_k,\xi_k)$ where $(v_k,\xi_k) \in \widehat{B}$.

Since u_k remains in a bounded of X, it is possible to extract a subsequence (still denoted (u_k)) which converges weakly to an element u of X, and like (3.28) we can get that

$$T(v_k,\xi_k) = u_k \longrightarrow u = T(v,\xi), \quad \text{Strongly in } X.$$

This implies that $\overline{T(\widehat{B})}^{X}$ is compact. Thus, we have proven the compactness of T.

• Let's prove now that $\exists C > 0$, such that

$$\forall (v,\xi) \in X \times [0,1] : v = T(v,\xi) \Rightarrow \|v\|_{\overrightarrow{p}(\cdot)} \leq C.$$

We have for $v \in X$ such that $v = T(v,\xi)$ meaning that

$$\text{for all } \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega): \ \sum_{i=1}^{N}\int_{\Omega}|\partial_i v|^{p_i(x)-1}\partial_i v \partial_i \varphi \, dx = \xi\left(\int_{\Omega} f_n \varphi \, dx + \sum_{i=1}^{N}\int_{\Omega}(|g|+|v|+|\partial_i v|)^{p_i(x)-1}\varphi \, dx\right). \tag{3.29}$$

Choosing $\varphi = v$ in (3.29), and using Young's inequality, Hölder inequality, Lemma 2.1, and (2.7), we get for all $\varepsilon > 0$:

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|\partial_i v|^{p_i(x)} \, dx &\leq 2\|f_n\|_{p_i'(\cdot)}\|v\|_{p_i(\cdot)} + \varepsilon\sum_{i=1}^{N}\int_{\Omega}(|g_n|^{p_i(x)} + |v|^{p_i(x)} + |\partial_i v|^{p_i(x)}) \, dx + C(\varepsilon)\sum_{i=1}^{N}\int_{\Omega}|v|^{p_i(x)} \, dx \\ &\leq c\|f_n\|_{p_i'(\cdot)}\|v\|_{\overrightarrow{p}(\cdot)} + \varepsilon\sum_{i=1}^{N}\int_{\Omega}(|g_n|^{p_i(x)} + |v|^{p_i(x)} + |\partial_i v|^{p_i(x)}) \, dx + C(\varepsilon)\sum_{i=1}^{N}\int_{\Omega}|v|^{p_i(x)} \, dx.\end{aligned} \tag{3.30}$$

By choosing $\varepsilon = \frac{1}{2}$, and the use of (3.1), (3.2), and the fact that $v \in L^{p_+(\cdot)}(\Omega)$, we obtain that

$$\|v\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \leq c\left(1 + \|v\|_{\overrightarrow{p}(\cdot)}\right). \tag{3.31}$$

By using separation of cases ($\|v\|_{\overrightarrow{p}(\cdot)} > 1$ and $\|v\|_{\overrightarrow{p}(\cdot)} \leq 1$), we can easily get from (3.31) that, $\exists C > 0$ independent of n such that

$$\|v\|_{\overrightarrow{p}(\cdot)} \leq C. \tag{3.32}$$

Since it is clear that $T(v,0) = 0$.

Then, the conditions for Leray-Schauder's fixed point Theorem were met. So, the operator $T_0 : X \longrightarrow X$ such that $T_0(u) = T(u, 1)$ accepts a fixed point. Therefore, the proof of Lemma 3.5 was completed. □

3.1.1. *A Priori Estimates.*

Lemma 3.6. *Let $\{u_n\}$ be the sequence of approximating solutions of (3.4) in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$. Assume f, g, p_i, $i \in \{1, \dots, N\}$ be restricted as in Theorem 3.2. Then, there exists $C > 0$ such that*

$$\|u_n\|_{\vec{p}(\cdot)} \leq C. \tag{3.33}$$

Moreover,

$$\partial_i u_n \longrightarrow \partial_i u \quad a.e. \quad in\ \overline{\Omega}, \quad i \in \{1, \dots, N\}. \tag{3.34}$$

Proof. By choosing $\varphi = u_n$ in (3.4), we get that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)}\, dx = \int_\Omega f_n u_n\, dx + \sum_{i=1}^{N} \int_\Omega (|g| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\, u_n\, dx.$$

By using the same way as proof (3.32), we easily get (3.33).

Now, (3.33) implies that, there exists a subsequence (still denoted by (u_n)) and $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega. \tag{3.35}$$

We put

$$\Delta_n = \sum_{i=1}^{N} \Delta_{n,i},$$

where

$$\Delta_{n,i} = \int_\Omega \left(|\partial_i u_n|^{p_i(x)-2} \partial_i u_n - |\partial_i u|^{p_i(x)-2} \partial_i u \right) (\partial_i u_n - \partial_i u)\, dx.$$

Let us first prove that,

$$\lim_{n\to+\infty} \Delta_n = 0. \tag{3.36}$$

We have

$$\Delta_n = \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx - \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx.$$

By choosing $\varphi = u_n - u$ in (3.4), and using (3.35), the facts that $\|f_n\|_{p_i'(\cdot)} < \infty$, $\|(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\|_{p_i'(\cdot)} < \infty$, we obtain that

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i(x)-2} \partial_i u_n (\partial_i u_n - \partial_i u)\, dx = 0. \tag{3.37}$$

By (3.35) and the fact that $\||\partial_i u|^{p_i(x)-2} \partial_i u\|_{p_i'(\cdot)} < \infty$, we obtain

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-2} \partial_i u (\partial_i u_n - \partial_i u)\, dx = 0. \tag{3.38}$$

Combining (3.37) and (3.38) yields (3.36).

From (2.2) we get that

$$\Delta_{n,i} > 0,\ i \in \{1, \dots, N\}. \tag{3.39}$$

Then, by (3.39) and (3.36) we obtain

$$\Delta_{n,i} \to 0,\ \text{strongly in } L^1(\Omega),\ i \in \{1, \dots, N\}. \tag{3.40}$$

By extracting a subsequence (still denoted by (u_n)), we conclude that

$$\Delta_{n,i} \longrightarrow 0 \quad \text{a.e. in } \Omega,\ i \in \{1, \dots, N\}. \tag{3.41}$$

So there is a subset $\Omega' \subset \Omega$ where $|\Omega'| = 0$, and $\forall x \in \Omega - \Omega'$

$$|\partial_i u(x)| < \infty, \text{ and } \Delta_{n,i} \to 0$$

By (3.41), we get

$$\Delta_{n,i}(x) \leq \phi(x), \tag{3.42}$$

for some functions ϕ.
Let's prove the existence of a function ψ such that

$$\big|\partial_i u_n(x)\big| \leq \psi(x). \tag{3.43}$$

By (3.42) and (2.2), we obtain

$$\begin{cases} \phi(x) \geq c\left((|\partial_i u_n| - |\partial_i u|)^{p_-^-} - 1\right), & \text{if } p_i(x) \geq 2, \\ \phi(x) \geq c'\left(\frac{|\partial_i u_n| - |\partial_i u|}{|\partial_i u_n| + |\partial_i u| + 1}\right)^2, & \text{if } 1 < p_i(x) < 2. \end{cases} \tag{3.44}$$

Then, (3.44) implies (3.43).
We proceed by contradiction to prove that

$$\partial_i u_n(x) \longrightarrow \partial_i u(x) \text{ in } \Omega - \Omega'. \tag{3.45}$$

Suppose there exists $a \in \Omega - \Omega'$ such that $\lim\limits_{n \to +\infty} \partial_i u_n(a) \neq \partial_i u(a)$.
So, Bolzano Weierstrass Theorem implies that

$$\partial_i u_n(a) \longrightarrow \eta \in \mathbb{R}. \tag{3.46}$$

By passing to the limit in $\Delta_{n,i}(a)$ when $n \longrightarrow +\infty$ and using (3.46), we obtain

$$\left(|\eta|^{p_i(a)-2}\eta - |\partial_i u(a)|^{p_i(a)-2}\partial_i u(a)\right)(\eta - \partial_i u(a)) = 0, \tag{3.47}$$

By (3.47) and (2.2), we conclude that $\eta = \partial_i u(a)$. This gives us (3.34). □

3.2. **Proof of the Theorem 3.2 :** By (3.34) and (3.35), we get that

$$|\partial_i u_n|^{p_i(x)-2}\partial_i u_n \longrightarrow |\partial_i u|^{p_i(x)-2}\partial_i u \text{ a.e. in } \Omega,\ i \in \{1, \dots, N\}. \tag{3.48}$$

By (3.33) we obtain that

$$\int_\Omega \left||\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right|^{p_i'(x)} dx = \int_\Omega |\partial_i u_n|^{p_i(x)} dx \leq c,\ i \in \{1, \dots, N\}. \tag{3.49}$$

Then, (3.49) and (2.3) implies that

$$\left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right) \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega),\ i \in \{1, \dots, N\}. \tag{3.50}$$

From Young's inequality and that $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we get $\forall \varepsilon > 0$

$$\begin{aligned} \int_\Omega \left||\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right| dx &= \int_\Omega |\partial_i u_n|^{p_i(x)-1} dx \\ &\leq C(\varepsilon) + \varepsilon \int_\Omega |\partial_i u_n|^{p_i(x)} dx \\ &\leq C(\varepsilon) + \varepsilon c = C'(\varepsilon). \end{aligned} \tag{3.51}$$

So, we conclude that

$$\left(|\partial_i u_n|^{p_i(x)-2}\partial_i u_n\right) \in L^1(\Omega),\ i \in \{1, \dots, N\}. \tag{3.52}$$

So, through (3.48), (3.52), and (3.50), and Vitali's Theorem [Lemma 3.4. in [8]], we derive, $\forall i \in \{1,\dots,N\}$

$$|\partial_i u_n|^{p_i(x)-2}\partial_i u_n \longrightarrow |\partial_i u|^{p_i(x)-2}\partial_i u \ , \text{ Strongly in } L^1(\Omega). \tag{3.53}$$

Now, through (3.34) and (3.35), we obtain that

$$(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} \longrightarrow (|g| + |u| + |\partial_i u|)^{p_i(x)-1} \quad \text{a.e. in } \Omega. \tag{3.54}$$

From another side, since g_n, u_n, $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we obtain, $\forall i \in \{1,\dots,N\}$

$$\begin{aligned}\int_\Omega \left|(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\right|^{p_i'(x)} dx &= \int_\Omega (|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)} dx \\ &\le c\int_\Omega \left(|g_n|^{p_i(x)} + |u_n|^{p_i(x)} + |\partial_i u_n|^{p_i(x)}\right) dx \le C. \end{aligned} \tag{3.55}$$

Then, (3.55) and (2.3) implies that, $\forall i \in \{1,\dots,N\}$

$$(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} \text{ uniformly bounded in } L^{p_i'(\cdot)}(\Omega). \tag{3.56}$$

Like the proof of (3.52) with the note that g_n, u_n, $\partial_i u_n \in L^{p_i(\cdot)}(\Omega)$, we can obtain, $\forall i \in \{1,\dots,N\}$

$$\left((|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1}\right) \in L^1(\Omega). \tag{3.57}$$

Then, from (3.57), (3.54), and (3.56), and Vitali's Theorem, we obtain that, $\forall i \in \{1,\dots,N\}$

$$(|g_n| + |u_n| + |\partial_i u_n|)^{p_i(x)-1} \longrightarrow (|g| + |u| + |\partial_i u|)^{p_i(x)-1} \text{ Strongly in } L^1(\Omega). \tag{3.58}$$

So through this, we can easily pass to the limit in (3.4). Thus Theorem 3.2 has been proven.

4. Conclusion

In this paper, we prove the existence of distributional solutions for a class of nonlinear anisotropic elliptic equations with variable exponents and nonlinearity linking the solution u and its partial derivatives $\partial_i u$, $i = 1,\dots,N$. We relied on proving the existence of a sequence of approximate solutions for a suitable approximate problem, thanks to Leray-Schauder's fixed-point theorem, and then achieving strong L^1-convergence to the desired solution.

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