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Symmetries and conservation laws of the Berger metric on a squashed three-sphere

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Abstract

In this work, we obtain Noether, Lie, and Killing symmetries of the Lagrangian of the Berger metric on a squashed three-sphere. With the help of the result of Noether's theorem, we have presented the expressions for conservation laws corresponding to all Noether symmetries.

Keywords. Berger metric, Squashed three-sphere, Noether symmetry, Killing symmetry.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

In this paper, we are interested in considering a symmetry analysis and conservation laws for the Berger metric of the squashed sphere. Assuming local coordinate (θ, ψ, t), the Berger metric of the squashed sphere is given (in Euler coordinates) by [11]:

$$4ds^2 = a^2(dt + \cos\theta d\psi)^2 + d\theta^2 + \sin^2\theta d\psi^2. \tag{1.1}$$

In this expression, the squashing parameter is one, corresponding to the round sphere, where a is the radius of the sphere, and the angles ψ and t obey the periodic identifications $\psi \to \psi + 2\psi$, $t \to t + 4\psi$, while the range of θ is $[0, \psi]$.

Today, recent geometric studies have mainly focused on pseudo-Riemannian geometry. Conservation laws of four-dimensional non-reductive homogeneous spaces were investigated in [6], while conformal Einstein pp-wave as quantum solutions were considered in [14]. Particular symmetry analysis of conservation laws on pseudo-Reimannian manifolds was also studied in several cases. Lie symmetries of the wave equation on conformally flat spaces were studied in [2]. Computation of partially invariant solutions for the Einstein Walker manifolds studied in [15]. Symmetry analysis of a fourth-order nonlinear diffusion and the Buckmaster equation studied through [3, 19]. The invariance of the Perturbed mKdVKS Equation was investigated in [13]. Conservation laws of a universal KP-like equation in $2+1$ dimensions are also considered in [1].

The purpose of this paper is to analyze the Noether, Lie, and Killing symmetries for a class of pseudo-Riemannian spaces, namely the squashed three-sphere ones equipped with metric (1.1). We give all the possible Noether, Lie, and Killing symmetries. After the Noether symmetries are determined, we will use a simple and essential way to determine the conservation laws of the Euler-Lagrange equations through a corollary of the Noether theorem [16]. Of course, this theorem depends on the availability of a Lagrangian and related Noether symmetries that keep the integral action invariant. Recently, there has been work [8] in which the relationship of Noether symmetry with the Killing vectors of some specific spacetimes is mentioned. In [4, 5, 7], Lie point symmetries and conservation laws are calculated. In another article [21], the authors propose a theorem based on which the Lie symmetries of geodesic equations in a Riemannian space depends on collineations of the metric. In the article [9], the author tries to analyze the symmetries of the charged squashed Kaluza-Klein black hole metric.

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This paper is organized in the following way. Section 2 is devoted to obtain data about the isometries of the metric from a study of the Noether symmetries associated with the corresponding natural Lagrangian. In section 3, we have collected some results related to Killing vector symmetry on the spaces under consideration. In the last section, we have presented the expressions for conservation laws corresponding to all Noether symmetries.

2. Noether symmetry

Metric (1.1), together with some choices for the different fields of supergravity background, can be coupled to a manifold of matter multiplets. The squashed 3-sphere appears as the spatial section of the frozen mixmaster universe and quantum field theory is explored on this space-time [17]. The squashed sphere can be considered as a space of harmonic spinors (the null space of the Dirac operator) on a manifold. Indeed, it is a good illustration of the fact that the number of harmonic spinors is not a topological invariant of the manifold, but depends on the particular metric too [18]. Homogeneously squashed sphere has been widely used in scalar quantum field theory.

The Lagrangian is practically a function that determines the dynamics (equations of motion) and symmetries of a dynamical system. The Lagrangian function, L, for a system is defined to be the difference between the kinetic and potential energies expressed as a function of positions and velocities. Let M is a Riemannian manifold with dimension n, equipped with the g metric. In a local coordinate $\mathbf{x} = (x^1, \dots, x^n)$, the corresponding Lagrangian for metric (1.1) is obtained using the following formula

$$L(s, x^\mu, \dot{x}^\mu) = \sum_{\mu,\nu} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu, \qquad \mu, \nu = 1, \cdots, 4, \tag{2.1}$$

where $g_{\mu\nu}(\mathbf{x})$ is a smooth function and $(g_{\mu\nu}(\mathbf{x}))$ is a positive definite matrix. Hence, the Lagrangian for metric (1.1) is

$$L = \frac{1}{4}a^2(\dot{t} + \cos\theta\dot{\psi})^2 + \frac{1}{4}\dot{\theta}^2 + \frac{1}{4}\sin\theta^2\dot{\psi}^2, \tag{2.2}$$

where the dot represents the derivative with respect to arc length s. First, we will determine Lie symmetries, and then discuss Noether symmetries. The Euler-Lagrange (geodesic) equations associated with the natural Lagrangian for metric (1.1) are

$$\begin{aligned}
&\frac{a^2}{2}(-\ddot{t} + \sin\theta\dot{\theta}\dot{\psi} - \cos\theta\ddot{\psi}) = 0,\\
&\frac{1}{2}(-a^2(\dot{t} - \cos\theta\dot{\psi})\sin\theta\dot{\psi} + \sin\theta\cos\theta(\dot{\psi})^2 - \ddot{\theta}) = 0,\\
&\frac{-a^2}{2}\cos\theta\ddot{t} - (\frac{-a^2}{2}\sin\theta\cos\theta\dot{\psi} - \frac{a^2}{2}(\dot{t} - \cos\theta\dot{\psi})\sin\theta + \sin\theta\cos\theta\dot{\psi})\dot{\theta} - (\frac{a^2}{2}\cos^2\theta + \frac{1}{2}\sin^2\theta)\ddot{\psi} = 0.
\end{aligned} \tag{2.3}$$

A generator of Lie symmetry as a Lie group of transformation keeps the system invariant. Calculations in the Lie point symmetries method for (2.3) create a system of partial differential equations which the total number of these equations is 68. After solving the obtained equations, the basis of Lie point symmetries for squashed three-sphere is three-dimensional, with the following bases:

$$\mathbf{v}_1 = \partial_s, \qquad \mathbf{v}_2 = \partial_\theta, \qquad \mathbf{v}_3 = \partial_t. \tag{2.4}$$

We show that notifications about metric isometries can be fully recovered by studying the Noether symmetries related to the corresponding natural Lagrangian, L. In general, in a local coordinate $\mathbf{x} = (x^1, x^2, \cdots, x^n)$ assume that

$$\mathbf{v} = \xi(s, \mathbf{x})\partial_s + \sum_\nu \eta^\nu(s, \mathbf{x})\partial_{x^\nu}, \tag{2.5}$$

is a vector field, that belongs to tangent space $T_{\mathbf{x}}M$. The expression

$$\mathbf{v}^{[1]} = \mathbf{v} + \sum_\nu \left(\eta^\nu_{,s} + \eta^\nu_{,s\mu}\dot{x}^\mu - \xi_{,s}\dot{x}^\nu - \xi_{,s}\dot{x}^\mu\dot{x}^\nu\right)\partial_{\dot{x}^\nu}, \tag{2.6}$$

is the first prolongation of the vector field (2.5) (see [12]). In this case, $\mathbf{v}$ is a Noether symmetry of the Lagrangian L, if there exists a gauge function, $A(s,\mathbf{x})$, such that

$$\mathbf{v}^{[1]}L+(D_s\xi)L=D_sA, \tag{2.7}$$

where

$$D_s=\partial_s+\sum_{\mu}\dot{x}^{\mu}\partial_{x^{\mu}}. \tag{2.8}$$

Using Eq. (2.8), a system of PDEs is obtained by solving which we determine the coefficients of the infinitesimal generators (2.5) (see [5]).

In case the squashed sphere of dimension three equipped with metric (1.1), $D_sA=0$. According to (2.5), the correspondence vector field of Lagrangian (2.2) is

$$\mathrm{v}=\xi(s,\theta,\psi,t)\partial_s+\tau(s,\theta,\psi,t)\partial_t+\eta(s,\theta,\psi,t)\partial_\theta+\zeta(s,\theta,\psi,t)\partial_\psi. \tag{2.9}$$

The first prolongation of Eq. (2.9) is as

$$\begin{aligned}\mathrm{v}^{[1]}=\mathrm{v}&+\left(\tau_s+\tau_\psi\dot{\psi}+\tau_\theta\dot{\theta}+\tau_t\dot{t}-\left[\xi_s+\xi_\psi\dot{\psi}+\xi_\theta\dot{\theta}+\xi_t\dot{t}\right]\dot{t}\right)\partial_{\dot{t}}\\&+\left(\eta_s+\eta_\psi\dot{\psi}+\eta_\theta\dot{\theta}+\eta_t\dot{t}-\left[\xi_s+\xi_\psi\dot{\psi}+\xi_\theta\dot{\theta}+\xi_t\dot{t}\right]\dot{\theta}\right)\partial_{\dot{\theta}}\\&+\left(\zeta_s+\zeta_\psi\dot{\psi}+\zeta_\theta\dot{\theta}+\zeta_t\dot{t}-\left[\xi_s+\xi_\psi\dot{\psi}+\xi_\theta\dot{\theta}+\xi_t\dot{t}\right]\dot{\psi}\right)\partial_{\dot{\psi}}\end{aligned} \tag{2.10}$$

which is obtained from formula (2.6), then substituting Lagrangian (2.2), and first prolonged vector field (2.10), into Eq. (2.7) assuming $A(s,\theta^{\mu})=0$, we find the following system of 20 partial differential equations:

$$\begin{aligned}-\frac{1}{4}\xi_t a^2&=-\frac{1}{4}\xi_\theta a^2=-\frac{1}{2}a^2\xi_\theta\cos\theta=-f_s=-\frac{1}{4}\xi_t=-\frac{1}{4}\xi_\theta\\&=-\frac{1}{4}\xi_\psi=-\frac{1}{2}a^2\xi_t\cos\theta-\frac{1}{4}\xi_\psi a^2=\frac{1}{2}\eta_s-f_\theta=\frac{1}{2}\eta_\theta-\frac{1}{4}\xi_s\\&=-\frac{1}{4}\xi_s a^2+\frac{1}{2}\zeta_t a^2\cos\theta+\frac{1}{2}a^2\tau_t=\frac{1}{4}\xi_\psi\cos^2\theta-\frac{1}{4}\xi_\psi a^2\cos^2\theta-\frac{1}{4}\xi_\psi\\&=+\frac{1}{2}\eta_t+\frac{1}{2}a^2\tau_\theta+\frac{1}{2}\zeta_\theta a^2\cos\theta=-f_t+\frac{1}{2}a^2\tau_s+\frac{1}{2}\zeta_s a^2\cos\theta\\&=-\frac{1}{4}\xi_\theta-\frac{1}{4}\xi_\theta a^2\cos^2\theta+\frac{1}{4}\xi_\theta\cos^2\theta\\&=-\frac{1}{4}\xi_t a^2\cos^2\theta-\frac{1}{4}\xi_t-\frac{1}{2}a^2\xi_\psi\cos\theta+\frac{1}{4}\xi_t\cos^2\theta\\&=+\frac{1}{2}a^2\tau_\theta\cos\theta-\frac{1}{2}\zeta_\theta\cos^2\theta+\frac{1}{2}a^2\cos^2\theta\zeta_\theta+\frac{1}{2}\eta_\psi+\frac{1}{2}\zeta_\theta\\&=-f_\psi+\frac{1}{2}a^2\tau_s\cos\theta+\frac{1}{2}\zeta_s+\frac{1}{2}a^2\zeta_s\cos^2\theta-\frac{1}{2}\zeta_s\cos^2\theta\\&=-\frac{1}{2}a^2\xi_s\cos\theta-\frac{1}{2}\zeta_t\cos^2\theta+\frac{1}{2}\zeta_\psi a^2\cos\theta+\frac{1}{2}\eta a^2\sin\theta+\frac{1}{2}\zeta_t a^2\cos^2\theta+\frac{1}{2}\zeta_t+\frac{1}{2}a^2\tau_t\cos\theta+\frac{1}{2}\tau_\psi a^2\\&=-\frac{1}{4}\xi_s+\frac{1}{2}\zeta_\psi-\frac{1}{2}a^2\eta\cos\theta\sin\theta-\frac{1}{2}\cos^2\theta\zeta_\psi+\frac{1}{2}\tau_\psi a^2\cos\theta\\&-\frac{1}{4}\xi_s a^2\cos^2\theta+\frac{1}{2}\zeta_\psi a^2\cos^2\theta+\frac{1}{2}\eta\cos\theta\sin\theta+\frac{1}{4}\xi_s\cos^2\theta=0.\end{aligned}$$

Solving this system of equations gives us the Noether symmetries Lie group associated with the Berger metric (1.1) possesses a Lie algebra generated by (2.6), whose coefficients are as the follows:

$$\begin{aligned}
&\xi = C_1,\\
&\eta = C_3 \sin\psi + C_4 \cos\psi,\\
&\tau = \frac{-C_3 \cos\psi + C_4 \sin\psi}{\sin\theta} + C_5,\\
&\zeta = \frac{C_3 \cos\psi - C_4 \sin\psi}{\tan\theta} + C_6,\\
&f = C_2,
\end{aligned} \tag{2.11}$$

where $C_i \in \mathbb{R}$, $i = 1, ..., 6$. Now, we can find the Noether symmetries of metric (1.1). The infinitesimal generator of Noether symmetries associated with the Berger metric (1.1) has five dimensions with the following bases

$$\begin{aligned}
&\mathbf{v}_1 := \partial_s,\\
&\mathbf{v}_2 := \partial_\psi,\\
&\mathbf{v}_3 := \partial_t,\\
&\mathbf{v}_4 := -\frac{\cos\psi\partial_t}{\sin\theta} + \sin\psi\partial_\theta + \frac{\cos\psi\cos\theta\partial_\psi}{\sin\theta},\\
&\mathbf{v}_5 := \frac{\sin\psi\partial_t}{\sin\theta} + \cos\psi\partial_\theta + \frac{\sin\psi\cos\theta\partial_\psi}{\sin\theta},
\end{aligned} \tag{2.12}$$

It should be noted that $\mathbf{v}_1$ is not a Killing vector, and $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ form the basis of a decomposable four-dimensional Lie algebra $\mathfrak{g}$ of isometries as $\mathfrak{su}(2, \mathbb{R}) \times \mathfrak{u}(1)$. There exists a basis $\{\vartheta_1, \cdots \vartheta_5\}$ of $\mathfrak{g}$ such that the non-zero Lie brackets are

$$[\vartheta_1, \vartheta_3] = \vartheta_4, \quad [\vartheta_1, \vartheta_4] = -\vartheta_3, \quad [\vartheta_3, \vartheta_4] = \vartheta_1.$$

Using the Lie algebra structure $\mathfrak{g}$, it is possible to study the geometric concepts (such as curvature concepts) of this space algebraically and independent of the coordinates.

3. Killing vector symmetry

Let M be an n-dimensional pseudo-Riemannian manifold with metric g. Geodesic equations as local space-time coordinates $\theta = (\theta^1, \dots, \theta^n)$, construct a system of nonlinear second-order ODEs

$$\ddot{\theta}^i + \sum_{j,k} \Gamma^i_{jk} \dot{\theta}_j \dot{\theta}_k = 0, \qquad 1 \le i \le n, \tag{3.1}$$

in which Γ^i_{jk} represents the Christoffel elements and "·" shows derivative concerning arc length s. Assume the nonlinear second-order system of n ODEs (3.1) as follows:

$$S_i(s, \theta, \theta^{(1)}, \theta^{(2)}) = 0, \qquad 1 \le i \le n, \tag{3.2}$$

where $\theta^{(t)}, 1 \le\ t \le 2$, is the t-th order derivative concerning s. Assume that the action of a one-parameter Lie group of transformations on the space (s, θ) is given as the following relation:

$$\bar{s} \mapsto s + \epsilon\xi(s, \theta), \qquad \bar{\theta}^\beta \mapsto \theta^\beta + \epsilon\eta^\beta(s, \theta), \tag{3.3}$$

where $\beta = 1, 2, \dots, n$, with associated infinitesimal generator

$$\mathbf{v} = \sigma(s, \theta)\partial_s + \sum_\beta \varphi^\beta \partial_{\theta^\beta}. \tag{3.4}$$

The second-order prolongation of vector field (3.4) is given by

$$\mathbf{v}^{[2]} = \mathbf{v} + \sum_\beta \left[\varphi^\beta_{,(1)}(s, \theta, \theta^{(1)})\partial_{\theta^\beta_{,(1)}} + \varphi^\beta_{,(2)}(s, \theta, \dots, \theta^{(2)})\partial_{\theta^\beta_{,(2)}} \right], \tag{3.5}$$

TABLE 1. Killing Lie algebra for metric (1.1).

$[\vartheta_i, \vartheta_j]$	ϑ_1	ϑ_2	ϑ_3	ϑ_4
ϑ_1	0	0	0	ϑ_1
ϑ_2	0	0	0	0
ϑ_3	0	0	0	0
ϑ_4	$-\vartheta_1$	0	0	0

where

$$\varphi^{\beta}_{,(1)} = D\varphi^{\beta}_{,(0)} - \theta^{\beta}_{,(1)} D\sigma, \qquad \varphi^{\beta}_{,(2)} = D\varphi^{\beta}_{,(1)} - \theta^{\beta}_{,(2)} D\sigma, \tag{3.6}$$

are the prolongation coefficients, and $\varphi^{\alpha}_{,(0)} = \varphi^{\alpha}(s, \theta)$, and D is the total derivative operator.

If the system (3.2) has the invariance condition under the one-parameter Lie group of transformations (3.3) then we have the invariance criteria [10]. Therefore, the vector field $\mathbf{v}$ is a Killing vector symmetry of (3.2) iff

$$\mathbf{v}^{[2]} S_i|_{S_i=0} = 0. \tag{3.7}$$

Solving (3.7), we obtain the determining equations as a system of partial differential equations. If these determining equations have solutions, then these solutions will be the Killing vector symmetry of (3.2).

Theorem 3.1. *The infinitesimal generators of Killing vector symmetry regarding Eq.* (1.1) *are four-dimensional, with the following bases:*

$$\vartheta_1 = \partial_s, \quad \vartheta_2 = \partial_\psi, \quad \vartheta_3 = \partial_t, \quad \vartheta_4 = s\partial_s.$$

Proof. The non-zero elements of the Christoffel symbols regarding to the metric (1.1) are

$$\begin{aligned} \Gamma^{\theta}_{\psi\psi} &= -\Gamma^{\psi}_{\theta\psi} = \frac{1}{4}(a^2 - 1)\sin\theta\cos\theta, \\ \Gamma^{\theta}_{\psi t} &= -\Gamma^{\psi}_{\theta t} = -\Gamma^{t}_{\theta\psi} = \frac{1}{8}a^2 \sin\theta. \end{aligned} \tag{3.8}$$

Replace the non-zero components (3.8) in (3.1). We earn the folloing geodesic equations for the metric (1.1):

$$\begin{aligned} S_1 =& \ddot{\theta} + \frac{1}{4}(a^2 - 1)\dot{\psi}\dot{\psi}\sin\theta\cos\theta + \frac{1}{4}a^2\dot{\psi}\dot{t}\sin\theta, \\ S_2 =& \ddot{\psi} - \frac{1}{2}(a^2 - 1)\dot{\theta}\dot{\psi}\sin\theta\cos\theta - \frac{1}{4}a^2\dot{\theta}\dot{t}\sin\theta, \\ S_3 =& \ddot{t} - \frac{1}{4}a^2\dot{\theta}\dot{\psi}\sin\theta. \end{aligned} \tag{3.9}$$

According to (3.4), the infinitesimal generator corresponding to the Killing vector symmetry of the system will be:

$$\mathbf{v} = \xi(s, \theta, \psi, t)\partial_s + \eta^1(s, \theta, \psi, t)\partial_\theta + \eta^2(s, \theta, \psi, t)\partial_\psi + \eta^3(s, \theta, \psi, t)\partial_t. \tag{3.10}$$

Applying the prolongation Equation (3.5) and also, the invariance condition (3.7), we have

$$\xi = C_1 s + C_2, \quad \eta^1 = 0, \quad \eta^2 = C_4, \quad \eta^3 = C_3.$$

We find Killing vector symmetries. □

The infinitesimal generators of Killing vector symmetry associated with metric (1.1) satisfy the following commutator tables:

4. Conservation Law regarding to Eq. (1.1)

A conservation law corresponding to a given system of differential equations is an expression

$$Div\rho = 0,$$

which becomes zero for all solutions $u = f(x)$ of the system. Here Div is divergence and

$$\rho = (\rho_1(x, u^{(n)}), \cdots, \rho_m(x, u^{(n)})),$$

is an m-tuple of smooth functions of θ, u. Let

$$X = \sum_{i=1}^{m} \xi^i(x, u)\partial_{x^i} + \sum_{j=1}^{q} \phi_j(x, u)\partial_{u^j},$$

is an infinitesimal generator for Noether symmetries and

$$Q_\alpha(x, u) = \phi_\alpha - \sum_{i=1}^{m} \xi^i u_i^\alpha,$$

the corresponding characteristic of X. By Noether's theorem, $Q = (Q_1, \cdots, Q_q)$ is the characteristic of conservation law for $E(L) = 0$; namely

$$Div\rho = Q.E(L),$$

is a conservation law for $E(L) = 0$. Section 2 states Noether symmetries. For arbitrary ξ, η, τ and ζ we have the following expression for ρ

$$\rho = \frac{1}{2}\eta\dot{\theta} + \zeta\left(\frac{1}{2}a^2(\dot{t} + \cos\theta\dot{\psi})\cos\theta + \frac{1}{2}\sin\theta^2\dot{\psi}\right) + \frac{1}{2}\tau a^2(\dot{t} + \cos\theta\dot{\psi})$$
$$+ \xi\left(\frac{1}{4}a^2(\dot{t} + \cos\theta\dot{\psi})^2 - \frac{1}{4}\dot{\theta}^2 + \frac{1}{4}\sin\theta^2\dot{\psi}^2 - \dot{\psi}\big(\frac{1}{2}a^2(\dot{t} + \cos\theta\dot{\psi})\cos\theta + \frac{1}{2}\sin\theta^2\dot{\psi}\big) - \frac{1}{2}\dot{t}a^2(\dot{t} + \cos\theta\dot{\psi})\right).$$

Now, we can obtain the corresponding conservation law for each Noether symmetry. The nonzero conservation laws for Noether symmetries are as follows:

1) For Noether symmetry infinitesimal generator ∂_s, the expression ρ is

$$\rho = \frac{1}{4}a^2(\dot{t} + \cos\theta\dot{\psi})^2 - \frac{1}{4}\dot{\theta}^2 + \frac{1}{4}\sin\theta^2\dot{\psi}^2 - \dot{\psi}\left(\frac{1}{2}a^2(\dot{t} + \cos\theta\dot{\psi})\cos\theta + \frac{1}{2}\sin\theta^2\dot{\psi}\right) - \frac{1}{2}\dot{t}a^2(\dot{t} + \cos\theta\dot{\psi}).$$

Thus, the conservation law is

$$\left(-\frac{1}{2}\dot{\psi}a^2\cos\theta - \frac{1}{2}a^2\dot{t}\right)\ddot{t} + \left(-\frac{1}{2}a^2(\dot{t} + \cos\theta\dot{\psi})\sin\theta\dot{\psi} + \frac{1}{2}\sin\theta\dot{\psi}^2\cos\theta\right.$$
$$\left.- \dot{\psi}\left(-\frac{1}{2}a^2\sin\theta\dot{\psi}\cos\theta - \frac{1}{2}a^2(\dot{t} + \cos\theta\dot{\psi})\sin\theta + \sin\theta\dot{\psi}\cos\theta\right) + \frac{1}{2}a^2\sin\theta\dot{\psi}\dot{t}\right)\dot{\theta}$$
$$- \frac{1}{2}\dot{\theta}\ddot{\theta} + \left(-\dot{\psi}\left(\frac{1}{2}a^2\cos\theta^2 + \frac{1}{2}\sin\theta^2\right) - \frac{1}{2}a^2\cos\theta\dot{t}\right)\ddot{\psi} = 0.$$

2) For ∂_ψ, the expression ρ is

$$\rho = \frac{1}{2}a^2\cos\theta\dot{t} + \frac{1}{2}a^2\cos\theta^2\dot{\psi} + \frac{1}{2}\dot{\psi} - \frac{1}{2}a^2\dot{\psi}\cos\theta^2.$$

Thus, the conservation law is

$$\frac{1}{2}a^2\cos\theta\ddot{t} + \left(-\frac{1}{2}a^2\sin\theta\dot{t} - a^2\sin\theta\dot{\psi}\cos\theta + \sin\theta\dot{\psi}\cos\theta\right)\dot{\theta} + \left(\frac{1}{2}a^2\cos\theta^2 + \frac{1}{2} - \frac{1}{2}\cos\theta^2\right)\ddot{\psi} = 0.$$

3) For ∂_t, the expression ρ is

$$\rho = \frac{1}{2}a^2(\dot{t} + \cos\theta\dot{\psi}).$$

Thus, the conservation law is

$$\frac{1}{2}a^2\ddot{t} - \frac{1}{2}a^2\sin\theta\dot{\psi}\dot{\theta} + \frac{1}{2}a^2\cos\theta\ddot{\psi} = 0.$$

4) For $-\frac{\cos\psi\partial t}{\sin\theta} + \sin\psi\partial\theta + \frac{\cos\psi\cos\theta\partial\psi}{\sin\theta}$, the expression ρ is

$$\rho = \frac{1}{2}\frac{1}{\sin\theta}\Big(\sin\psi\dot{\theta}\sin\theta + \cos\psi a^2\cos\theta^2\dot{t} + \cos\psi a^2\cos\theta^3\dot{\psi} + \cos\psi\dot{\psi}\cos\theta - \cos\psi\dot{\psi}\cos\theta^3 - \cos\psi a^2\dot{t} - \cos\psi a^2\dot{\psi}\cos\theta\Big).$$

Thus, the conservation law is

$$\begin{aligned}
&\frac{\left(\frac{1}{2}\cos\psi a^2\cos\theta^2 - \frac{1}{2}\cos\psi a^2\right)\ddot{t}}{\sin\theta} + \Big(\frac{1}{\sin\theta}(\frac{1}{2}\sin\psi\dot{\theta}\cos\theta - \cos\psi\sin\theta a^2\cos\theta\dot{t} - \frac{3}{2}\cos\psi\sin\theta a^2\cos\theta^2\dot{\psi}\\
&-\frac{1}{2}\cos\psi\dot{\psi}\sin\theta + \frac{3}{2}\cos\psi\dot{\psi}\cos\theta^2\sin\theta + \frac{1}{2}\cos\psi a^2\sin\theta\dot{\psi})\\
&-\frac{1}{\sin\theta^2}((\frac{1}{2}\sin\psi\dot{\theta}\sin\theta + \frac{1}{2}\cos\psi a^2\cos\theta^2\dot{t} + \frac{1}{2}\cos\psi a^2\cos\theta^3\dot{\psi} + \frac{1}{2}\cos\psi\dot{\psi}\cos\theta - \frac{1}{2}\cos\psi\dot{\psi}\cos\theta^3\\
&-\frac{1}{2}\cos\psi a^2\dot{t} - \frac{1}{2}\cos\psi a^2\dot{\psi}\cos\theta)\cos\theta)\Big)\dot{\theta} + \frac{1}{2}\sin\psi\ddot{\theta} + \frac{1}{\sin\theta}((\frac{1}{2}\cos\psi\dot{\theta}\sin\theta - \frac{1}{2}\sin\psi a^2\cos\theta^2\dot{t}\\
&-\frac{1}{2}\sin\psi a^2\cos\theta^3\dot{\psi} - \frac{1}{2}\sin\psi\dot{\psi}\cos\theta + \frac{1}{2}\sin\psi\dot{\psi}\cos\theta^3 + \frac{1}{2}\sin\psi a^2\dot{t} + \frac{1}{2}\sin\psi a^2\dot{\psi}\cos\theta)\dot{\psi})\\
&+\frac{1}{\sin\theta}((\frac{1}{2}\cos\psi a^2\cos\theta^3 + \frac{1}{2}\cos\psi\cos\theta - \frac{1}{2}\cos\psi\cos\theta^3 - \frac{1}{2}\cos\psi a^2\cos\theta)\ddot{\psi}) = 0,
\end{aligned}$$

5) For $\frac{\sin\psi\partial t}{\sin\theta} + \cos\psi\partial\theta + \frac{\sin\psi\cos\theta\partial\psi}{\sin\theta}$, the expression ρ is

$$\begin{aligned}
\rho = \frac{1}{2}\frac{1}{\sin\theta}\Big(&\cos\psi\dot{\theta}\sin\theta - \sin\psi a^2\cos\theta^2\dot{t} - \sin\psi a^2\cos\theta^3\dot{\psi} - \sin\psi\dot{\psi}\cos\theta\\
&+\sin\psi\dot{\psi}\cos\theta^3 + \sin\psi a^2\dot{t} + \sin\psi a^2\dot{\psi}\cos\theta\Big).
\end{aligned}$$

Thus, the conservation law is

$$\begin{aligned}
&\frac{\left(-\frac{1}{2}\sin\psi a^2\cos\theta^2 + \frac{1}{2}\sin\psi a^2\right)\ddot{t}}{\sin\theta} + \Big(\frac{1}{\sin\theta}(\frac{1}{2}\cos\psi\dot{\theta}\cos\theta + \sin\psi\sin\theta a^2\cos\theta\dot{t} + \frac{3}{2}\sin\psi\sin\theta a^2\cos\theta^2\dot{\psi}\\
&-\frac{1}{2}\sin\psi\dot{\psi}\sin\theta - \frac{3}{2}\sin\psi\dot{\psi}\cos\theta^2\sin\theta - \frac{1}{2}\sin\psi a^2\sin\theta\dot{\psi}) - \frac{1}{\sin\theta^2}((\frac{1}{2}\cos\psi\dot{\theta}\sin\theta - \frac{1}{2}\sin\psi a^2\cos\theta^2\dot{t}\\
&-\frac{1}{2}\sin\psi a^2\cos\theta^3\dot{\psi} - \frac{1}{2}\sin\psi\dot{\psi}\cos\theta + \frac{1}{2}\sin\psi\dot{\psi}\cos\theta^3 + \frac{1}{2}\sin\psi a^2\dot{t} + \frac{1}{2}\sin\psi a^2\dot{\psi}\cos\theta)\cos\theta)\Big)\dot{\theta} + \frac{1}{2}\cos\psi\ddot{\theta}\\
&+\frac{1}{\sin\theta}((-\frac{1}{2}\sin\psi\dot{\theta}\sin\theta - \frac{1}{2}\cos\psi a^2\cos\theta^2\dot{t} - \frac{1}{2}\cos\psi a^2\cos\theta^3\dot{\psi} - \frac{1}{2}\cos\psi\dot{\psi}\cos\theta + \frac{1}{2}\cos\psi\dot{\psi}\cos\theta^3\\
&+\frac{1}{2}\cos\psi a^2\dot{t} + \frac{1}{2}\cos\psi a^2\dot{\psi}\cos\theta)\dot{\psi}) + \frac{1}{\sin\theta}((-\frac{1}{2}\sin\psi a^2\cos\theta^3\\
&-\frac{1}{2}\sin\psi\cos\theta + \frac{1}{2}\sin\psi\cos\theta^3 + \frac{1}{2}\sin\psi a^2\cos\theta)\ddot{\psi}) = 0.
\end{aligned}$$

5. Conclusion

In this research, we obtained some practical and useful information by finding and analyzing the symmetries of the Euler-Lagrange equations of the Berger metric on a squashed three-sphere. In this case, the Euler-Lagrange equations were determined and we observed that we have a 5-dimensional Lie algebra of Noether symmetries. Then we determined the Lie algebra structure of isometries as $\mathfrak{su}(2,\mathbb{R}) \times \mathfrak{u}(1)$ which provides the possibility to investigate the geometric concepts (such as curvature concepts) of the Berger metric on a squashed three-sphere algebraically. For the obtained Noether symmetries, the related conservation laws are also calculated.

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