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Investigation of highly dispersive solitons for the concatenation model with power law nonlinearity using the improved modified extended tanh-function method

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Abstract

This research examines the phenomenon of optical solitons in the framework of the dispersive concatenation model, which incorporates three established models: the Lakshmanan-Porsezian-Daniel equation (LPDE), the Hirota equation (HE), and the nonlinear Schrödinger equation (NLSE). This model describes the soliton transmission dynamics across transcontinental and transoceanic dynamics. The model provided is situated within the context of nonlinear optics, a branch of optics that deals with optical phenomena in materials where the response of the medium to light is nonlinear. The equation appears to be a generalized model that combines several well-known equations from nonlinear optics. These equations often emerge as simplified descriptions of specific nonlinear effects in various optical systems. They capture phenomena like self-focusing, self-phase modulation, and soliton propagation, among others. The improved modified extended tanh scheme (IMETS) is utilized to derive solitons and other solutions for the investigated model. Many types of solutions are extracted with the help of the IMETS. These solutions include dark, bright, and singular solitons, as well as Weierstrass elliptic and singular periodic solutions. The nature of the extracted solutions is illustrated by introducing both 2D and 3D graphical representations and setting the parameters with appropriate values.

Keywords. Bright solitons, Dark solitons, Exact solutions, Nonlinear partial differential equations.

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1. Introduction

A variety of physical systems such as fluid dynamics, optics and plasma physics are modeled with the aid of NLSE which characterizes the progression of a function with complex values that represent the wave envelope. The nonlinearity in the equation stems from the fact that the wave speed depends on the wave amplitude. The nonlinear Schrödinger equation is of utmost importance in examining optical solitons. As these solitons move through a nonlinear medium, they can maintain their velocities and shapes. NLSEs can be utilized for the purpose of designing and enhancing optical fiber systems for a wide range of applications, including telecommunications and optical computing.

Several recent studies have focused on soliton solutions including the Gerdjikov-Ivanov equation [13, 15], the Radhak–rishnan-Kundu-Lakshmanan equation [3, 9], the Kundu-Eckhaus equation [12], HE [4], LPDE [10], and others. The dispersive concatenation model is a composite of three separate mathematical models: the NLSE with fifth order dispersion, the LPDE, and the HE model. Each of these models elucidates distinct physical phenomena associated with nonlinear waves, including optical solitons, rogue waves, and other nonlinear waves. By examining the concatenation model, scientists can gain a more comprehensive understanding of the characteristics of nonlinear waves and develop new solutions to the associated physical dilemmas. Researching the concatenation model is a potential

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area of study that can provide new insights into the mechanics of nonlinear waves [1, 2, 5–7, 14]. The dispersive concatenation model reads as [23]:

$$i\varphi_t + a\varphi_{xx} + b|\varphi|^2\varphi - i\delta_1(\sigma_1\varphi_{xxx} + \sigma_2|\varphi|^2\varphi_x) + \delta_2[\sigma_3\varphi_{xxxx} + \sigma_4|\varphi|^2\varphi_{xx} + \sigma_5|\varphi|^4\varphi + \sigma_6|\varphi_x|^2\varphi + \sigma_7(\varphi_x)^2\varphi^* + \sigma_8\varphi^*_{xx}\varphi^2] - i\delta_3[\sigma_9\varphi_{xxxxx} + \sigma_{10}|\varphi|^2\varphi_{xxx} + \sigma_{11}|\varphi|^4\varphi_x + \sigma_{12}\varphi\varphi_x\varphi^*_{xx} + \sigma_{13}\varphi^*\varphi_x\varphi_{xx} + \sigma_{14}\varphi\varphi^*_x\varphi_{xx} + \sigma_{15}\varphi^*_x(\varphi_x)^2] = 0. \tag{1.1}$$

Here, $\varphi(x,t)$ describes the wave profile. The temporal evolution is denoted by the first term. Chromatic dispersion and Self-Phase modulation are denoted by a and b, respectively. All parameters σ_j for $j = 1, 2, 3, ..., 15$ are constants. Eq. (1.1) converts to the classical NLSE when $\delta_1 = \delta_2 = \delta_3 = 0$ while HE model is generated in case of $\delta_1 \neq 0$ and $\delta_2 = \delta_3 = 0$. For $\delta_1 = \delta_3 = 0$ with $\delta_2 \neq 0$, Eq. (1.1) reduces to LPDE.

The proposed model was studied by employing the enhanced Kudryashov's approach to get straddled, bright, and singular optical solitons. This algorithm not only provides a nuanced understanding of the various soliton types but also highlights the occurrence of singular solitons that exhibit unique characteristics. In addition, the Riccati equation expansion approach was applied to obtain dark solitons in addition to singular solitons. Furthermore, the Weierstrass' expansion scheme was implemented to encompass bright, singular solitons [23]. In this work, the improved modified extended tanh function method is implemented to study Eq. (1.1). This method is based on the extended Riccati equation which can produce various and novel types of solutions for the investigated model. This solutions including bright, dark and singular solitons. In addition, singular periodic, rational, exponential, and Weierstrass elliptic solutions can be raised.

2. Integration scheme of IMETS

This part briefly discusses the IMETF approach [8, 16]. Considering the next NLPDE

$$F(\varphi, \varphi_t, \varphi_x, \varphi_{xx}, \varphi_{tx}, ...) = 0. \tag{2.1}$$

The subsequent steps should be performed to handle Eq. (2.1) using the IMETF scheme:
Step(1): Utilizing the subsequent transformation:

$$\varphi(x,t) = \varphi(\xi), \tag{2.2}$$

where $\xi = x - c\, t,$. After that, Eq. (2.1) can be expressed as:

$$G(\varphi, \varphi', \varphi'', \varphi''', ...) = 0. \tag{2.3}$$

Step(2): The resulted NLODE's solutions can be represented as:

$$\varphi(\xi) = \sum_{j=0}^{k} a_j \Upsilon^j(\xi) + \sum_{j=-1}^{-k} b_{-j}\Upsilon^j(\xi), \tag{2.4}$$

where $\Upsilon(\xi)$ satisfy the following differential equation:

$$\Upsilon'(\xi) = \sqrt{d_0 + d_1\Upsilon(\xi) + d_2\Upsilon^2(\xi) + d_3\Upsilon^3(\xi) + d_4\Upsilon^4(\xi)}. \tag{2.5}$$

Step(3): k can be evaluated by applying the balance rule on Eq. (2.3).
Step(4): The NLODE is transformed into nonlinear algebraic equation (NLAE) by inserting Eqs. (2.4) and (2.5) into Eq. (2.3).
Step(5): A system of NLAE is then generated by collecting and equating the coefficients of $\Upsilon^j(\xi)$ with zero. The resultant system can be handled to evaluate a_j, b_j, and c using Mathematica packages.
Step(6): One can get many solutions for Eq. (2.1) by substituting the determined constants and the general solutions of (2.5) into Eq. (2.4).

Comparing with other techniques such as extended direct algebraic method [17], simple ansatze method [18], the Unified method [20], the modified and extended rational expansion method [19], the extended (G'/G^2)-expansion technique [11], the variational principle method [21, 22], this method gives various types of solutions such as bright, dark, and singular solitons. In addition, other mathematical solutions such as exponential, Weierstrass elliptic, rational type and singular periodic solutions can be extracted. However, when N, which is evaluated via the balance rule, is large, it will give a more complex system that is difficult to solve.

3. Mathematical Analysis

The purpose of this section is to obtain exact solutions to Eq. (1.1) in the format indicated below:

$$\begin{aligned} \varphi(x,t) &= \Re(\xi)e^{i\psi(x,t)},\\ \xi &= x-ct,\\ \psi(x,t) &= -\kappa x+\omega t+\theta_0. \end{aligned} \tag{3.1}$$

Here κ denotes the soliton frequency while θ_0 denotes the phase constant. Wave number is denoted by ω while c denotes the soliton speed. By inserting Eq. (3.1) into Eq. (1.1) and then dividing into real (Re) and imaginary (IM) portions, we obtain:
Re parts:

$$\begin{aligned} &(\delta_2\sigma_3-5\delta_3\kappa\sigma_9)\Re^{(4)}(\xi)+(10\delta_3\kappa^3\sigma_9-6\delta_2\kappa^2\sigma_3-3\delta_1\kappa\sigma_1+a)\Re^{''}(\xi)\\ &+(\delta_1\kappa^3\sigma_1+\delta_2\kappa^4\sigma_3-\delta_3\kappa^5\sigma_9-a\kappa^2-\omega)\Re(\xi)\\ &+(2\delta_3\kappa\sigma_{12}-2\delta_3\kappa\sigma_{13}-2\delta_3\kappa\sigma_{14}-\delta_3\kappa\sigma_{15}+\delta_2\sigma_6+\delta_2\sigma_7)\Re(\xi)\Re^{'2}(\xi)\\ &+(-3\delta_3\kappa\sigma_{10}-\delta_3\kappa\sigma_{12}-\delta_3\kappa\sigma_{13}+\delta_3\kappa\sigma_{14}+\delta_2\sigma_4+\delta_2\sigma_8)\Re^2(\xi)\Re^{''}(\xi)+(\delta_2\sigma_5-\delta_3\kappa\sigma_{11})\Re^5(\xi)\\ &+(\delta_3\kappa^3\sigma_{10}+\delta_3\kappa^3\sigma_{12}+\delta_3\kappa^3\sigma_{13}-\delta_3\kappa^3\sigma_{14}-\delta_3\kappa^3\sigma_{15}-\delta_1\kappa\sigma_2-\delta_2\kappa^2\sigma_4+\delta_2\kappa^2\sigma_6-\delta_2\kappa^2\sigma_7\\ &-\delta_2\kappa^2\sigma_8+b)\Re^3(\xi)=0, \end{aligned} \tag{3.2}$$

and the IM parts:

$$\begin{aligned} &-\delta_3\sigma_9\Re^{(5)}(\xi)+(10\delta_3\kappa^2\sigma_9-4\delta_2\kappa\sigma_3-\delta_1\sigma_1)\Re^{'''}(\xi)+(-5\delta_3\kappa^4\sigma_9+4\delta_2\kappa^3\sigma_3+3\delta_1\kappa^2\sigma_1-2a\kappa-c)\Re^{'}(\xi)\\ &-\delta_3\sigma_{15}\Re^{'3}(\xi)-(\delta_3\sigma_{12}+\delta_3\sigma_{13}+\delta_3\sigma_{14})\Re(\xi)\Re^{'}(\xi)\Re^{''}(\xi)-\delta_3\sigma_{10}\Re^2(\xi)\Re^{'''}(\xi)\\ &+(3\delta_3\kappa^2\sigma_{10}-\delta_3\kappa^2\sigma_{12}+3\delta_3\kappa^2\sigma_{13}-\delta_3\kappa^2\sigma_{14}-\delta_3\kappa^2\sigma_{15}-2\delta_2\kappa\sigma_4-2\delta_2\kappa\sigma_7\\ &+2\delta_2\kappa\sigma_8-\delta_1\sigma_2)\Re^2(\xi)\Re^{'}(\xi)-\delta_3\sigma_{11}\Re^4(\xi)\Re^{'}(\xi)=0. \end{aligned} \tag{3.3}$$

Differentiating Eq. (3.2) with regard to ξ, we get

$$\begin{aligned} &(\delta_2\sigma_3-5\delta_3\kappa\sigma_9)\Re^{(5)}(\xi)+(10\delta_3\kappa^3\sigma_9-6\delta_2\kappa^2\sigma_3-3\delta_1\kappa\sigma_1+a)\Re^{'''}(\xi)\\ &(\delta_1\kappa^3\sigma_1+\delta_2\kappa^4\sigma_3-\delta_3\kappa^5\sigma_9-a\kappa^2-\omega)\Re^{'}(\xi)\\ &(2\delta_3\kappa\sigma_{12}-2\delta_3\kappa\sigma_{13}-2\delta_3\kappa\sigma_{14}-\delta_3\kappa\sigma_{15}+\delta_2\sigma_6+\delta_2\sigma_7)\Re^{'3}(\xi)+(2\delta_3\kappa\sigma_{12}-6\delta_3\kappa\sigma_{13}\\ &-2\delta_3\kappa\sigma_{14}-2\delta_3\kappa\sigma_{15}+2\delta_2\sigma_6+2\delta_2\sigma_7-6\delta_3\kappa\sigma_{10}+2\delta_2\sigma_4+2\delta_2\sigma_8)\Re(\xi)\Re^{'}(\xi)\Re^{''}(\xi)\\ &+(-3\delta_3\kappa\sigma_{10}-\delta_3\kappa\sigma_{12}-\delta_3\kappa\sigma_{13}+\delta_3\kappa\sigma_{14}+\delta_2\sigma_4+\delta_2\sigma_8)\Re^2(\xi)\Re^{'''}(\xi)+5(\delta_2\sigma_5-\delta_3\kappa\sigma_{11})\Re^4(\xi)\Re^{'}(\xi)\\ &+3(\delta_3\kappa^3\sigma_{10}+\delta_3\kappa^3\sigma_{12}+\delta_3\kappa^3\sigma_{13}-\delta_3\kappa^3\sigma_{14}-\delta_3\kappa^3\sigma_{15}-\delta_1\kappa\sigma_2-\delta_2\kappa^2\sigma_4\\ &+\delta_2\kappa^2\sigma_6-\delta_2\kappa^2\sigma_7-\delta_2\kappa^2\sigma_8+b)\Re^2(\xi)\Re^{'}(\xi)=0. \end{aligned} \tag{3.4}$$

Eqs. (3.3) and (3.4) will be similar under the following circumstances:

$$\begin{aligned} &\delta_2\sigma_3+\delta_3\sigma_9(1-5\kappa)=0,\\ &10\delta_3\kappa^2\sigma_9(\kappa-1)+a+2\sigma_3\delta_2\kappa(2-3\kappa)+\delta_1\sigma_1(1-3\kappa)=0,\\ &\delta_1\kappa^2\sigma_1(\kappa-3)+\delta_2\sigma_3\kappa^3(\kappa-4)-\delta_3\sigma_9\kappa^4(\kappa-5)+a\kappa^2-\omega+2a\kappa+c=0,\\ &2\delta_3\kappa\sigma_{12}-2\delta_3\kappa\sigma_{13}-2\delta_3\kappa\sigma_{14}+\delta_3\sigma_{15}(1-\kappa)+\delta_2\sigma_6+\delta_2\sigma_7=0,\\ &\delta_3\sigma_{12}(2\kappa+1)+\delta_3\sigma_{13}(1-6\kappa)+\delta_3\sigma_{14}(1-2\kappa)-2\delta_3\kappa\sigma_{15}+2\delta_2\sigma_6+2\delta_2\sigma_7-6\delta_3\kappa\sigma_{10}+2\delta_2\sigma_4+2\delta_2\sigma_8=0,\\ &\delta_3\sigma_{10}(1-3\kappa)-\delta_3\kappa\sigma_{12}-\delta_3\kappa\sigma_{13}+\delta_3\kappa\sigma_{14}+\delta_2\sigma_4+\delta_2\sigma_8=0, \end{aligned}$$

$$
\begin{aligned}
&5\delta_2\sigma_5+\delta_3\sigma_{11}(1-5\kappa)=0,\\
&3\delta_3\kappa^2\sigma_{10}(\kappa-1)+\delta_3\kappa^2\sigma_{12}(1-3\kappa)+3\delta_3\sigma_{13}\kappa^2(\kappa-1)+\delta_3\kappa^2\sigma_{14}(1-3\kappa)\delta_3\kappa^2\sigma_{15}(1-3\kappa)\\
&+\delta_1\sigma_2(1-3\kappa)+\delta_2\kappa\sigma_4(2-3\kappa)+3\delta_2\kappa^2\sigma_6+\delta_2\kappa\sigma_7(2-3\kappa)-\delta_2\kappa\sigma_8(3\kappa+2)+3b=0.
\end{aligned}
$$

The following form can now be used to rewrite Eq. (3.3):

$$
\begin{aligned}
&\Re^{(5)}(\xi)+\Delta_1\Re^{'''}(\xi)+\Delta_2\Re^{'}(\xi)+\Delta_3\Re^{'3}(\xi)+\Delta_4\Re(\xi)\Re^{'}(\xi)\Re^{''}(\xi)\\
&\qquad+\Delta_5\Re^2(\xi)\Re^{'''}(\xi)+\Delta_6\Re^2(\xi)\Re^{'}(\xi)+\Delta_7\Re^4(\xi)\Re^{'}(\xi)=0,
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
\Delta_1&=\frac{-1}{\delta_3\sigma_9}(10\delta_3\kappa^2\sigma_9-4\delta_2\kappa\sigma_3-\delta_1\sigma_1),\\
\Delta_2&=\frac{-1}{\delta_3\sigma_9}(-5\delta_3\kappa^4\sigma_9+4\delta_2\kappa^3\sigma_3+3\delta_1\kappa^2\sigma_1-2a\kappa-c),\\
\Delta_3&=\frac{\sigma_{15}}{\sigma_9},\\
\Delta_4&=\frac{1}{\sigma_9}(\sigma_{12}+\sigma_{13}+\sigma_{14}),\\
\Delta_5&=\frac{\sigma_{10}}{\sigma_9},\\
\Delta_6&=\frac{-1}{\delta_3\sigma_9}(3\delta_3\kappa^2\sigma_{10}-\delta_3\kappa^2\sigma_{12}+3\delta_3\kappa^2\sigma_{13}-\delta_3\kappa^2\sigma_{14}-\delta_3\kappa^2\sigma_{15}-2\delta_2\kappa\sigma_4-2\delta_2\kappa\sigma_7+2\delta_2\kappa\sigma_8-\delta_1\sigma_2),\\
\Delta_7&=\frac{\sigma_{11}}{\sigma_9}.
\end{aligned}
$$

Applying balance rule, we have $k=1$. Subsequently, the solutions of Eq. (3.5) can be represented as:

$$
\Re_1(\xi)=a_0+a_1\Upsilon+\frac{b_1}{\Upsilon}.
\tag{3.6}
$$

Using the procedures (2.2), (2.3), and (2.4) mentioned in the last part, the next results are derived:
Set 1. $g_0=g_1=g_3=0$

Result (I)

$$
a_0=0,\qquad b_1=0,\qquad \Delta_2=-\frac{2g_4\left(a_1^2\Delta_6+6\Delta_1g_4\right)\left(a_1^4\Delta_1\Delta_7+a_1^2\left(2\Delta_6-\Delta_1\left(\Delta_3-4\Delta_5\right)\right)g_4+12\Delta_1g_4^2\right)}{a_1^4\left(\left(\Delta_3-4\Delta_5\right)g_4-a_1^2\Delta_7\right)^2},
$$

$$
g_2=\frac{2g_4\left(a_1^2\Delta_6+6\Delta_1g_4\right)}{a_1^2\left(a_1^2\Delta_7-\Delta_3g_4+4\Delta_5g_4\right)},\qquad g_4=-\frac{2a_1^2\Delta_7}{\sqrt{\left(\Delta_3+2\Delta_4+6\Delta_5\right)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5}.
$$

Then, we derive

$$
\begin{aligned}
\varphi(x,t)=&\left\{\sqrt{\frac{24\Delta_1\Delta_7-2\Delta_6\left(\sqrt{\left(\Delta_3+2\Delta_4+6\Delta_5\right)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5\right)}{\left(\sqrt{\left(\Delta_3+2\Delta_4+6\Delta_5\right)^2-480\Delta_7}+3\Delta_3+2\Delta_4-2\Delta_5\right)\Delta_7}}\right.\\
&\left.\times\operatorname{sech}\left(\frac{2(ct-x)\sqrt{\frac{12\Delta_1\Delta_7}{\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5}-\Delta_6}}{\sqrt{\sqrt{\left(\Delta_3+2\Delta_4+6\Delta_5\right)^2-480\Delta_7}+3\Delta_3+2\Delta_4-2\Delta_5}}\right)\right\}\times e^{i\psi(x,t)},
\end{aligned}
\tag{3.7}
$$

$$\varphi(x,t)=\left\{\sqrt{\frac{24\Delta_1\Delta_7-2\Delta_6\left(\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^{2}-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5\right)}{\left(\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^{2}-480\Delta_7}+3\Delta_3+2\Delta_4-2\Delta_5\right)\Delta_7}}\right.$$
$$\left.\times\sec\left(\frac{2(ct-x)\sqrt{\Delta_6-\frac{12\Delta_1\Delta_7}{\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5}}}{\sqrt{\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^{2}-480\Delta_7}+3\Delta_3+2\Delta_4-2\Delta_5}}\right)\right\}\times e^{i\psi(x,t)}. \tag{3.8}$$

A bright solitary is denoted by Eq. (3.7) whereas Eq. (3.8) denotes a singular periodic solution.

Set 2. $g_1=g_3=0, g_0=\frac{g_2^2}{4g_4}$

Result (I)

$$a_0=0,\ b_1=0,\ g_4=-\frac{a_1^2\Delta_3g_2^2}{4\left(\Delta_2+\Delta_1g_2+4g_2^2\right)},$$
$$\Delta_6=\frac{a_1^2\Delta_3g_2^2-2a_1^2\Delta_4g_2^2-2a_1^2\Delta_5g_2^2+12\Delta_2g_4-72g_4g_2^2}{2a_1^2g_2},$$
$$\Delta_7=-\frac{g_4\left(a_1^2\Delta_3+2a_1^2\Delta_4+6a_1^2\Delta_5+120g_4\right)}{a_1^4}.$$

Then, we derive

$$\varphi(x,t)=\sqrt{2}\sqrt{\frac{\Delta_2+\Delta_1g_2+4g_2^2}{\Delta_3g_2}}\times\tanh\left(\frac{\sqrt{-g_2}(x-ct)}{\sqrt{2}}\right)\times e^{i\psi(x,t)}, \tag{3.9}$$

$$\varphi(x,t)=\sqrt{2}\sqrt{-\frac{\Delta_2+\Delta_1g_2+4g_2^2}{\Delta_3g_2}}\times\tan\left(\frac{\sqrt{g_2}(x-ct)}{\sqrt{2}}\right)\times e^{i\psi(x,t)}. \tag{3.10}$$

A dark solitary is denoted by Eq. (3.9) whereas Eq. (3.10) denotes a singular periodic solution.

Result (II)

$$a_0=0,\ g_4=-\frac{a_1g_2}{2b_1},\ b_1=\frac{\Delta_2+4g_2\left(\Delta_1+16g_2\right)}{8a_1\Delta_3g_2},$$
$$a_1=\frac{\left(\Delta_3+2\Delta_4+6\Delta_5\right)g_2-\sqrt{\left((\Delta_3+2\Delta_4+6\Delta_5)^{2}-480\Delta_7\right)g_2^2}}{4\Delta_7},$$
$$\Delta_6=\frac{8g_2\left(a_1b_1\left(\Delta_3-2\left(\Delta_4+\Delta_5\right)\right)+36g_2\right)-3\Delta_2}{4a_1b_1}.$$

Then, we derive

$$\varphi(x,t)=\frac{-1}{\sqrt{2}\sqrt{\Delta_3g_2}}\left\{\sqrt{\Delta_2+4g_2\left(\Delta_1+16g_2\right)}\times\coth\left(\sqrt{2}\sqrt{-g_2}(x-ct)\right)\right\}\times e^{i\psi(x,t)}, \tag{3.11}$$

$$\varphi(x,t)=\frac{1}{\sqrt{\Delta_3g_2}}\left\{\sqrt{-\frac{\Delta_2}{2}-2\Delta_1g_2-32g_2^2}\times\cot\left(\sqrt{2}\sqrt{g_2}(x-ct)\right)\right\}\times e^{i\psi(x,t)}. \tag{3.12}$$

Eq. (3.11) is a singular solitary solution whereas Eq. (3.12) represents a singular periodic solution.
Set (3). $g_2=g_4=0$

Result (I)

$$\begin{aligned}
&b_1 = 0,\ \Delta_7 = 0,\ \Delta_3 = \frac{1}{2}(-3)\left(\Delta_4 + 2\Delta_5\right),\\
&\Delta_1 = \frac{a_1^3\Delta_4 g_1 + 3a_1^3\Delta_5 g_1 + 3a_0^2 a_1\Delta_4 g_3 + 9a_0^2 a_1\Delta_5 g_3 + 45a_0 g_3^2}{3a_1 g_3},\\
&\Delta_2 = \frac{3a_1^4\left(\Delta_4 + 2\Delta_5\right) g_0 - a_0 a_1^3\Delta_4 g_1 + 3a_0^3 a_1\left(\Delta_4 + 4\Delta_5\right) g_3 - 9a_1^2 g_1 g_3 + 45a_0^2 g_3^2}{2a_1^2},\\
&g_3 = -\frac{1}{30}a_1\left(\sqrt{a_0^2\left(\Delta_4 + 4\Delta_5\right)^2 - 40\Delta_6} + a_0\left(\Delta_4 + 4\Delta_5\right)\right).
\end{aligned}$$

Then, we derive

$$\varphi(x,t) = \left\{a_0 + a_1\wp\left(\frac{(x-ct)\sqrt{-a_1\left(\sqrt{a_0^2\left(\Delta_4 + 4\Delta_5\right)^2 - 40\Delta_6} + a_0\left(\Delta_4 + 4\Delta_5\right)\right)}}{2\sqrt{30}}, \{h_2, h_3\}\right)\right\} \times e^{i\psi(x,t)}, \tag{3.13}$$

where $h_2 = -4g_1/g_3$ and $h_3 = -4g_0/g_3$. Eq. (3.13) denotes Weierstrass elliptic solution
Set (4). $g_0 = g_1 = g_2 = 0$
Result (I)

$$\begin{aligned}
&b_1 = 0,\ a_0 = -\frac{\sqrt{-\frac{\sqrt{\Delta_6^2 - 4\Delta_2\Delta_7} + \Delta_6}{\Delta_7}}}{\sqrt{2}},\ g_3 = \frac{4a_0 g_4}{a_1},\ g_4 = \frac{a_1^2\Delta_7}{3\left(\Delta_3 - 2\Delta_5\right)},\\
&\Delta_1 = \frac{-2a_1^2 a_0^2\Delta_7 - a_1^2\Delta_6 - 6a_0^2\Delta_5 g_4}{6g_4},\ \Delta_4 = -\frac{2\left(a_1^4\Delta_7 + 6a_1^2\Delta_5 g_4 + 90g_4^2\right)}{3a_1^2 g_4}.
\end{aligned}$$

Then, we derive

$$\varphi(x,t) = \left\{\frac{\sqrt{-\frac{\sqrt{\Delta_6^2 - 4\Delta_2\Delta_7} + \Delta_6}{\Delta_7}}\left(-2(x-ct)^2\left(\sqrt{\Delta_6^2 - 4\Delta_2\Delta_7} + \Delta_6\right) + 9\Delta_3 - 18\Delta_5\right)}{\sqrt{2}\left(2(x-ct)^2\left(\sqrt{\Delta_6^2 - 4\Delta_2\Delta_7} + \Delta_6\right) + 3\Delta_3 - 6\Delta_5\right)}\right\} \times e^{i\psi(x,t)}, \tag{3.14}$$

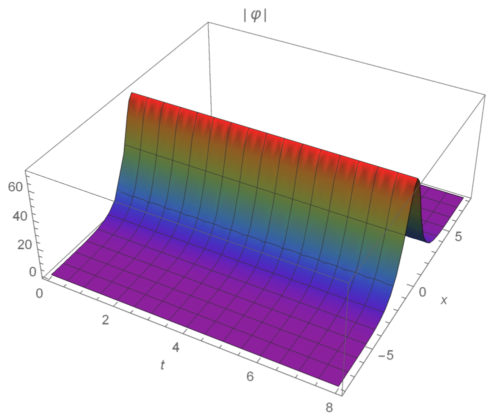
$$\varphi(x,t) = \left\{\sqrt{-\frac{\sqrt{\Delta_6^2 - 4\Delta_2\Delta_7} + \Delta_6}{2\Delta_7}} \times \left(-1 - 2e^{\sqrt{\frac{2}{3}}(x-ct)\sqrt{\frac{\sqrt{\Delta_6^2 - 4\Delta_2\Delta_7} + \Delta_6}{\Delta_3 - 2\Delta_5}}}\right)\right\} \times e^{i\psi(x,t)}. \tag{3.15}$$

A rational solution is derived by Eq. (3.14) whereas Eq. (3.15) denotes an exponential solution.

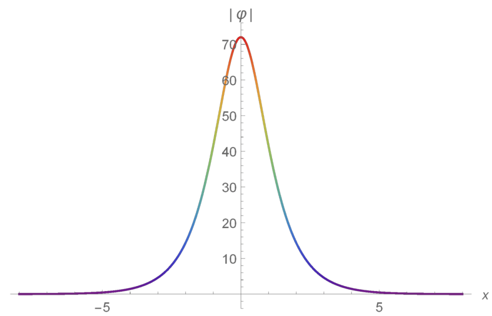
Set (5). $g_0 = g_1 = 0, g_4 = \frac{g_3^2}{4g_2}$

Result (I)

$$\begin{aligned}
&b_1 = 0,\quad a_0 = \frac{a_1 g_2}{g_3},\quad g_2 = -\frac{\left(\sqrt{\left(\Delta_3 + 2\Delta_4 + 6\Delta_5\right)^2 - 480\Delta_7} + \Delta_3 + 2\Delta_4 + 6\Delta_5\right) g_3^2}{8a_1^2\Delta_7},\\
&\Delta_1 = -\frac{g_2\left(4a_1^4\Delta_7 g_2^2 + a_1^2\Delta_4 g_2 g_3^2 + 5a_1^2\Delta_5 g_2 g_3^2 + 2a_1^2\Delta_6 g_3^2 + 15g_3^4\right)}{3g_3^4},\\
&\Delta_2 = \frac{g_2^2\left(a_1^4\Delta_7 g_2^2 + a_1^2\Delta_4 g_2 g_3^2 + 2a_1^2\Delta_5 g_2 g_3^2 - a_1^2\Delta_6 g_3^2 + 12g_3^4\right)}{3g_3^4}.
\end{aligned}$$



(A)



(B)

Figure 1. Bright solitary solution of Eq. (3.7).

Then, we derive dark solitary solution

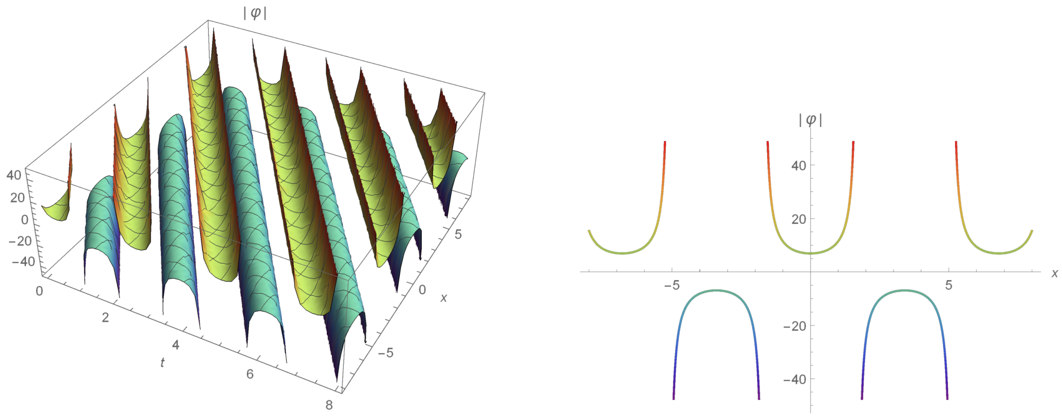
$$\varphi(x,t) = \Bigg\{ \frac{1}{4\sqrt{2}} \times \sqrt{-\frac{\left(\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5\right) g_3^2}{\Delta_7 g_4}} \times \left(\tanh\left(\frac{(x-ct)\sqrt{-\frac{\left(\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5\right) g_3^2}{a_1^2\Delta_7}}}{4\sqrt{2}} \right) + 1 \right) - \frac{\left(\sqrt{(\Delta_3+2\Delta_4+6\Delta_5)^2-480\Delta_7}+\Delta_3+2\Delta_4+6\Delta_5\right) g_3}{8a_1\Delta_7} \Bigg\} \times e^{i\psi(x,t)}. \tag{3.16}$$

4. 3D and 2D visualizations

In order to highlight the properties of the derived solutions, we present the two-dimensional and three-dimensional graphs of some solutions. Figure 1 shows a bright solitary solution of Eq. (3.7) with $c = 0,\ \Delta_1 = 2,\ \Delta_2 = 0.18,\ \Delta_3 = 0.25,\ \Delta_4 = 1,\ \Delta_5 = 0.285,\ \Delta_6 = -2,\ \Delta_7 = 0.001$. Singular periodic solutions of Eq. (3.8) are introduced in Figure 2 with $c = -2.95,\ \Delta_1 = -0.565,\ \Delta_3 = -0.025,\ \Delta_4 = -0.615,\ \Delta_5 = 0.04,\ \Delta_6 = 0.46,\ \Delta_7 = -0.02$. Figure 3 shows a singular solitary solution of Eq. (3.11) with $c = 0.02,\ \Delta_1 = -2,\ \Delta_2 = -2,\ \Delta_3 = -2,\ g_2 = -2$. Figure 4 shows a dark solitary solution of Eq. (3.16) with $c = -0.48,\ a_1 = -0.22,\ g_4 = 0.02,\ g_3 = -0.26,\ \Delta_3 = -0.15,\ \Delta_4 = 0.86,\ \Delta_5 = 0.1,\ \Delta_7 = -0.28$.

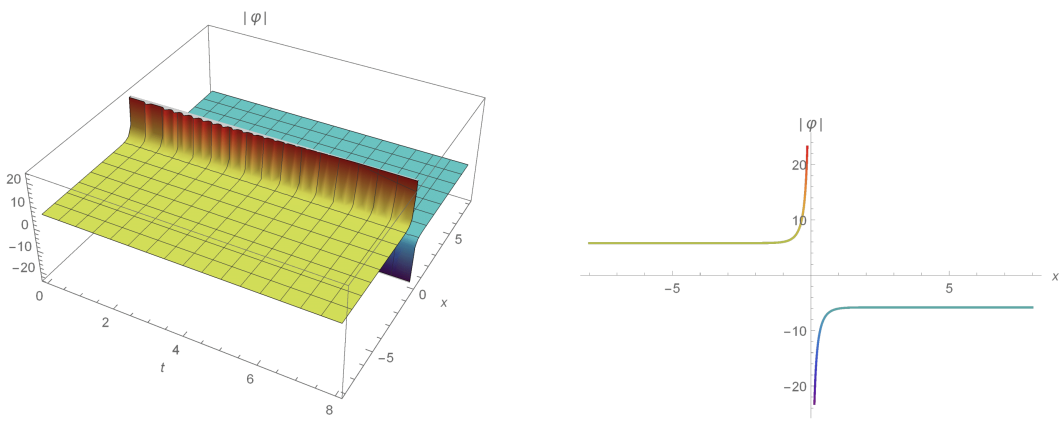
5. Discussion and Results

This paper successfully ventured and recovered optical soliton solutions to the dispersive concatenation model with linear chromatic dispersion and self-phase modulation. Earlier investigations have laid the groundwork by introducing concatenation models combining established equations such as the NLSE, LPDE model, and Sasa-Satsuma equation. We have extended these models to incorporate higher-order dispersive effects, introducing equations like the SHE and quintic-order NLSE. Studying was conducted with the aid of the improved modified extended tanh function method, various and novel solutions were raised. These solutions include bright, dark, and singular solitons. In addition, other mathematical solutions such as Weierstrass elliptic, exponential, rational and singular periodic solutions were constructed. 3D and 2D graphical representations were illustrated for some selected solutions to show the nature of the



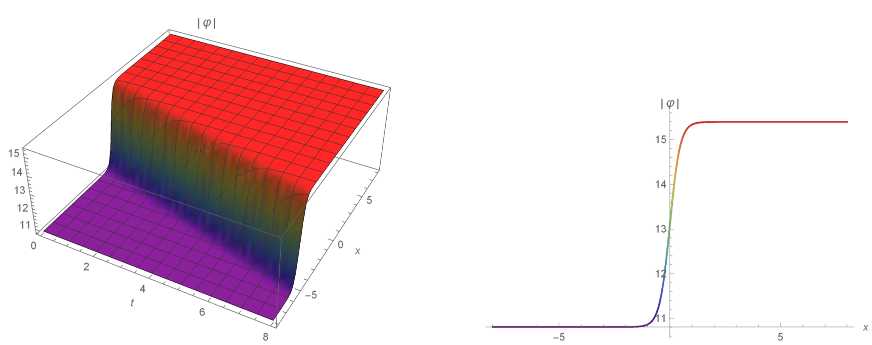
(A) (B)

FIGURE 2. Singular periodic solution of Eq. (3.8).



(A) (B)

FIGURE 3. Singular solitary solution of Eq. (3.11).



(A) (B)

FIGURE 4. Dark solitary solution of Eq. (3.16).

propagated waves. These solutions were presented by setting the parameters with appropriate values. The obtained solitons proved that a dedicated balance occurred between the non linear and dispersion terms. These solutions will help in the development of the communication industry as these types of solutions can propagate to very long distances maintaining their shapes and speeds.

6. Conclusion

The IMETS was successfully applied in this work to investigate dispersive optical solitons. Many solitons and other solutions were extracted. These solutions include bright, dark, and singular solitary solutions, Weierstrass elliptic and singular periodic solutions. Moreover, graphical representations in both 2D and 3D of some of the recovered solutions are presented to illustrate the characteristics of the propagating wave. These solutions provide an explanation for a wide range of fascinating and challenging physical phenomena due to the NLSE model's applicability in several scientific domains, including wave-guides and optical fibers. The retrieved solutions in this research study are novel, and the model was not previously investigated using the proposed methodology. The approach's success, convenience of use, and efficacy show the method's applicability for dealing with nonlinear optical problems. With all of these features, it will undoubtedly enrich the literature. The results are thus tremendously promising and lead to the avenues of further research in this arena. Later, the model will be studied with differential group delay followed by the consideration of the model with dispersion-flattened fibers. In addition, the stochastic model can be investigated to show the effect of the noise on the extracted solutions.

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