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A Chebyshev wavelet approach to the generalized time-fractional Burgers-Fisher equation

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Abstract

This work proposes a new method for obtaining the approximate solution of the time-fractional generalized Burgers-Fisher equation. The method's main idea is based on converting the nonlinear partial differential equation to a linear partial differential equation using the Picard iteration method. Then, the second kind Chebyshev wavelet collocation method is used to solve the linear equation obtained in the previous step. The technique is called the Chebyshev Wavelet Picard Method (CWPM). The proposed method successfully solves the time fractional generalized Burgers-Fisher equation. The obtained numerical results are compared with the exact solutions and with the solutions obtained using the Haar wavelet Picard method and the homotopy perturbation method.

Keywords. Numerical methods for wavelets, Fractional partial differential equations, Fractional derivatives and integrals, Picard iteration technique, Sylvester equation.

2010 Mathematics Subject Classification. 65T60, 35R11, 26A33, 65M70.

1. Introduction

Many physical phenomena are modeled using nonlinear equations in various fields of engineering and applied sciences. Fractional partial differential equations are used in the modeling of physical and chemical phenomena and applied sciences, for instance, oscillation of earthquakes [14], economics [6], signal processing [31], control theory [8], and solid mechanics [34].

However, a few methods are proposed to solve these equations, such as the variational iteration method (VIM) ([10, 38]), Adomian decomposition method (ADM) ([11–17]), homotopy analysis method [13], power series method [29], and Laplace transform method [32], which are often complicated and time-consuming. Considering these limitations, we introduce a reliable and effective method for solving these equations in this article.

The generalized Burgers-Fisher equation is one of the most important nonlinear partial differential equations which appears in various applications, such as heat conduction, fluid dynamics, sound waves in a viscous medium, shock wave formation, turbulence, traffic flow, and some other fields of applied sciences ([18–20, 23, 27, 30, 42]).

The generalized time-fractional Burgers-Fisher equation with the Caputo fractional derivative is:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au^\delta u_x - u_{xx} = bu(1-u^\delta), \qquad x \in [0,1], \quad t \geq 0, \quad 0 < \alpha \leq 1, \tag{1.1}$$

where a, b, and δ are constants. The exact solution for $\alpha = 1$ is given by [17] and [33] in the following form:

$$u(x,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}\left[x - \left(\frac{a^2 + b(1+\delta)^2)}{a(1+\delta)}\right)t\right]\right)\right)^{\frac{1}{\delta}}, \tag{1.2}$$

Many researchers have investigated analytical and numerical techniques to solve the generalized Burgers-Fisher equation. For example, Kumar et al. [21] used the discontinuous Legendre wavelet Galerkin method, Singh et al. [37]

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used the B-spline collocation method, Saeed et al. [35] used the CAS wavelet quasilinearization technique, Ismail et al. [17] used Adomian decomposition method, Rashidi et al. [33] employed the homotopy perturbation method (HPM), Nawaz et al. [28] applied the optimal homotopy asymptotic method, Umar Saeed and Mujeeb Rehman [36] employed Haar wavelet Picard method (HWPM), to obtain approximate solutions of the generalized Burgers-Fisher equation.

Wavelets are used widely to solve ordinary and partial differential equations; mainly, orthogonal wavelets are extensively used to approximate the numerical solutions of different types of fractional partial differential equations in different articles, e.g. [1–5, 7, 15, 16, 22, 24–26, 39]. Among them, the second-kind Chebyshev wavelet has gained much attention due to the following reasons: first, because of their useful properties ([9, 12, 43]), second, it handles better than other numerical methods in singularities because of the fast convergence of Chebyshev wavelet method, last but not least, unlike various numerical methods, it does not undergo from the instability problems.

In the present paper, we use a combination of second kind Chebyshev wavelet collocation and Picard iteration technique for the numerical solution of the generalized Burgers-Fisher equation with time-fractional derivative. We obtained an operational matrix for the fractional integral of a single Chebyshev wavelet in the Riemann-Liouville sense through shifted Chebyshev polynomials of the second kind. The numerical results show that the proposed method is very convenient for solving such problems.

2. Second kind Chebyshev wavelet

The second kind Chebyshev wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments k, n, m, t, which k can assume any positive integer, $n = 1, 2, 3, \dots, 2^{k-1}$, m is the degree of the second kind Chebyshev polynomials and t is the normalized time. They are defined on the interval $[1, 0)$ as:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{U}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ 0, & Otherwise, \end{cases} \tag{2.1}$$

where

$$\tilde{U}_m(t) = \sqrt{\frac{2}{\pi}} U_m(t), \qquad m = 0, 1, 2, \dots, M-1. \tag{2.2}$$

$U_m(t)$'s are the second kind Chebyshev polynomials of degree m, which are orthogonal with respect to the weight function $w(t) = \sqrt{1-t^2}$ on the interval $[-1, 1]$ and satisfy the following recursive formula:

$$\begin{aligned} &U_0(t) = 1, \qquad U_1(t) = 2t, \\ &U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \ \ m = 1, 2, 3, \dots. \end{aligned} \tag{2.3}$$

The weight function $\tilde{w}(t) = w(2t-1)$ has to be dilated and translated as $w_n(t) = w(2^k t - 2n + 1)$.

A function $f(x) \in \mathcal{L}^2(\mathbb{R})$ defined over $[0, 1)$ can be expressed by the second kind Chebyshev wavelets as:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n,m} \psi_{n,m}(x), \tag{2.4}$$

where $c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle$. If the infinite series in Eq. (2.4) is truncated, then it can be written as:

$$f(x) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) = C^T \Psi(x), \tag{2.5}$$

where the coefficient vector C and the second Chebyshev wavelet function vector $\Psi(x)$ are $m' = 2^{k-1}M$ column vectors. For simplicity, Eq. (2.5) can be written as:

$$f(x) \cong \sum_{i=1}^{m'} c_i \psi_i(x) = C^T \Psi(x), \tag{2.6}$$

where $c_i = c_{nm}$, $\psi_i(t) = \psi_{nm}(t)$. The index i can be determined by the relation $i = M(n-1)+m+1$; thus, we have:

$$C = [c_1, c_2, \dots, c_{m'}]^T, \tag{2.7}$$
$$\Psi(t) = [\psi_1, \psi_2, \dots, \psi_{m'}]^T.$$

By taking the collocation points $t_i = \frac{2i-1}{2^k M}$, $i = 1, 2, 3, \dots, 2^{k-1}M$, we define the second Chebyshev wavelet matrix $\Phi_{m'\times m'}$ as:

$$\Phi_{m'\times m'} = \left[\Psi\left(\frac{1}{2m'}\right), \Psi\left(\frac{3}{2m'}\right), \dots, \Psi\left(\frac{2m'-1}{2m'}\right)\right],$$

where $m' = 2^{k-1}M$. For example, when $M = 4$ and $k = 2$, the second Chebyshev wavelet matrix is expressed as:

$$\Phi_{8\times 8} = \begin{pmatrix} 1.5958 & 1.5958 & 1.5958 & 1.5958 & 0 & 0 & 0 & 0 \\ -2.3937 & -2.3937 & -2.3937 & -2.3937 & 0 & 0 & 0 & 0 \\ 1.9947 & 1.9947 & 1.9947 & 1.9947 & 0 & 0 & 0 & 0 \\ -0.5984 & -0.5984 & -0.5984 & -0.5984 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5958 & 1.5958 & 1.5958 & 1.5958 \\ 0 & 0 & 0 & 0 & -2.3937 & -2.3937 & -2.3937 & -2.3937 \\ 0 & 0 & 0 & 0 & 1.9947 & 1.9947 & 1.9947 & 1.9947 \\ 0 & 0 & 0 & 0 & -0.5984 & -0.5984 & -0.5984 & -0.5984 \end{pmatrix}.$$

In the same way, a function $u(x,t) \in \mathcal{L}^2([0,1]\times[0,1])$ can be approximated as:

$$u(x,t) = \Psi(x)^T U \Psi(t), \tag{2.8}$$

which U is an $m' \times m'$ matrix with $u_{ij} = \langle \psi_i(x), \langle u(x,t), \psi_j(t)\rangle\rangle$. We use the wavelet collocation method to determine the coefficients $u_{i,j}$.

Theorem 2.1 ([40])**.** *Let $f(x)$ be a second-order differentiable square-integrable function defined on $[0,1)$ with bounded second-order derivative, say $f''(x) \le B$ for some constant B, then*

(i) $f(x)$ can be expanded as an infinite sum of the second kind Chebyshev wavelets, and the series converges to $f(x)$ uniformly, that is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x),$$

where $c_{nm} = \langle f(x), \psi_{nm}(x)\rangle$.

(ii)

$$\sigma_{f,k,M} < \frac{\sqrt{\pi}B}{2^3}\left(\sum_{n=2^{k-1}+1}^{\infty}\frac{1}{n^5}\sum_{m=M}^{\infty}\frac{1}{(m-1)^4}\right)^{\frac{1}{2}},$$

where $\sigma_{f,k,M} = \left(\int_0^1 \left|f(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x)\right|^2 w_n(x)dx\right)^2.$

2.1. **The fractional integral of the second kind Chebyshev wavelet.** In this section, the fractional integral formula of the Chebyshev wavelet in the Riemann-Liouville sense is derived using the shifted second-kind Chebyshev polynomials U_m^*, which plays a vital role in dealing with the time fractional equations.

Theorem 2.2 ([41])**.** *The fractional integral of a Chebyshev wavelet defined on the interval $[0,1]$ with compact support $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]$ is given by*

$$I^\alpha\psi_{n,m}(x) = \begin{cases} 0, & x < \frac{n-1}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)}2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\sum_{r=0}^{m}\sum_{i=r}^{m}\sum_{j=0}^{r} T_{i,i-r}^{m,n,k}\frac{(-1)^j}{j+\alpha}\times C_r^j x^{r-j}\left(x-\frac{n-1}{2^{k-1}}\right)^{j+\alpha}, & \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, \\ \frac{1}{\Gamma(\alpha)}2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}\sum_{r=0}^{m}\sum_{i=r}^{m}\sum_{j=0}^{r} T_{i,i-r}^{m,n,k}\frac{(-1)^j}{j+\alpha}\times C_r^j x^{r-j}\left(\left(x-\frac{n-1}{2^{k-1}}\right)^{j+\alpha}-\left(x-\frac{n}{2^{k-1}}\right)^{j+\alpha}\right), & x > \frac{n}{2^{k-1}}, \end{cases} \tag{2.9}$$

where $T^{m,n,k}_{i,i-r} = (-1)^{m-r}2^{2i}2^{r(k-1)}(n-1)^{i-r}\left(\frac{\Gamma(m+i+2)}{\Gamma(m-i+1)\Gamma(2i+2))}\right)\left(\frac{i!}{(i-r)!r!}\right)$, $\quad C^j_r = \frac{r!}{j!(r-j)!}$.

For instance, in the case of $k=2$, $M=4$, $x=0.6$, $\alpha=0.9$, we obtain:

$$I^\alpha\Psi_{8\times1}(0.6) = \begin{pmatrix} 0.838817891721642 \\ 0.045706956934399 \\ 0.290734994150959 \\ 0.021626272477045 \\ 0.208881853762857 \\ -0.329813453309774 \\ 0.323368822612918 \\ -0.217309447751042 \end{pmatrix},$$

where $\Psi_{8\times1} = (\psi_{1,0}(x)\ \ \psi_{1,1}(x)\ \ \psi_{1,2}(x)\ \ \psi_{1,3}(x)\ \ \psi_{2,0}(x)\ \ \psi_{2,1}(x)\ \ \psi_{2,2}(x)\ \ \psi_{2,3}(x))^T$.

We can obtain the fractional order integration matrix $P^\alpha_{m'\times m'} = I^\alpha\psi_{n,m}(x)$ by substituting the collocation points in Eq. (2.9) as:

$$P^\alpha_{2^{k-1}M\times 2^{k-1}M} = \begin{pmatrix} I^\alpha\psi_{1,0}(x_1) & I^\alpha\psi_{1,0}(x_2) & \dots & I^\alpha\psi_{1,0}(x_{2^{k-1}M}) \\ I^\alpha\psi_{1,1}(x_1) & I^\alpha\psi_{1,1}(x_2) & \dots & I^\alpha\psi_{1,1}(x_{2^{k-1}M}) \\ \vdots & \vdots & \ddots & \vdots \\ I^\alpha\psi_{2^{k-1},M-1}(x_1) & I^\alpha\psi_{2^{k-1},M-1}(x_2) & \dots & I^\alpha\psi_{2^{k-1},M-1}(x_{2^{k-1}M}) \end{pmatrix},$$

In particular, if we fix $k=2$, $M=4$ and $\alpha=0.9$, then

$$P^{0.9}_{8\times8} = \begin{pmatrix} 0.1368 & 0.3678 & 0.5825 & 0.7885 & 0.8517 & 0.8165 & 0.7939 & 0.7771 \\ -0.2377 & -0.4452 & -0.3985 & -0.1245 & 0.0545 & 0.0337 & 0.0246 & 0.0194 \\ 0.2789 & 0.2423 & 0.0032 & 0.0615 & 0.2996 & 0.2783 & 0.2680 & 0.2612 \\ -0.2570 & -0.0232 & -0.0530 & -0.2259 & 0.0274 & 0.0148 & 0.0104 & 0.0081 \\ 0 & 0 & 0 & 0 & 0.1368 & 0.3678 & 0.5825 & 0.7885 \\ 0 & 0 & 0 & 0 & -0.2377 & -0.4452 & -0.3985 & -0.1245 \\ 0 & 0 & 0 & 0 & 0.2789 & 0.2423 & 0.0032 & 0.0615 \\ 0 & 0 & 0 & 0 & -0.2570 & -0.0232 & -.0530 & -0.2259 \end{pmatrix}.$$

2.2. **Operational matrix of fractional integration for boundary value problems.** We derive another operational matrix of fractional integration to solve the fractional boundary value problems. Let $\eta > 0$ and $g : [0,\eta] \to \mathbb{R}$ be a continuous function, then we put:

$$g(x)I^\alpha\psi_{n,m}(\eta) = v^{\alpha,\eta}. \tag{2.10}$$

We define a matrix V by using the collocation points $x_i = \frac{2i-1}{2^kM}$, $\ i = 1,2,\dots,2^{k-1}M$ in Eq. (2.10), we get:

$$V^{\alpha,\eta,g(x)}_{2^{k-1}M\times 2^{k-1}M} = \begin{pmatrix} g(x_1)I^\alpha\psi_{1,0}(\eta) & g(x_2)I^\alpha\psi_{1,0}(\eta) & \dots & g(x_{2^{k-1}M})I^\alpha\psi_{1,0}(\eta) \\ g(x_1)I^\alpha\psi_{1,1}(\eta) & g(x_2)I^\alpha\psi_{1,1}(\eta) & \dots & g(x_{2^{k-1}M})I^\alpha\psi_{1,1}(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_1)I^\alpha\psi_{2^{k-1},M-1}(\eta) & g(x_2)I^\alpha\psi_{2^{k-1},M-1}(\eta) & \dots & g(x_{2^{k-1}M})I^\alpha\psi_{2^{k-1},M-1}(\eta) \end{pmatrix}. \tag{2.11}$$

In particular, for $\eta=1$, $g(x)=x$, $\alpha=0.9$, $k=2$ and $M=4$, we get:

$$V^{0.9,1,x}_{8\times8} = \begin{pmatrix} 0.0374 & 0.1112 & 0.1870 & 0.2618 & 0.3366 & 0.4114 & 0.4862 & 0.5610 \\ -0.0083 & -0.0249 & -0.0416 & -0.0582 & -0.0748 & -0.0914 & -0.1080 & -0.1247 \\ 0.0125 & 0.0374 & 0.0623 & 0.0873 & 0.1112 & 0.1371 & 0.1621 & 0.1870 \\ -0.0033 & -0.0100 & -0.0166 & -0.0233 & -0.0299 & -0.0366 & -0.0432 & -0.0499 \\ 0.0125 & 0.0374 & 0.0623 & 0.0873 & 0.1122 & 0.1371 & 0.1621 & 0.1870 \\ -0.0083 & -0.0249 & -0.0416 & -0.0582 & -0.0748 & -0.0914 & -0.1080 & -0.1247 \\ 0.0042 & 0.0125 & 0.0208 & 0.0291 & 0.0374 & 0.0457 & 0.0540 & 0.0623 \\ -0.0033 & -0.0100 & -0.0166 & -0.0233 & -0.0299 & -0.0366 & -0.0432 & -0.0499 \end{pmatrix}.$$

3. Description of the proposed method

Here, we elaborate on the method of solving the time fractional generalized Burgers-Fisher equation. The first step is converting the fractional nonlinear partial differential equation to a linear equation using the Picard iteration technique. The second step is to solve the obtained linear equation by the second kind Chebyshev wavelet collocation method.

We apply the Picard iteration method to Eq. (1.1):

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} - \frac{\partial^2 u_r}{\partial x^2} = b u_r(1-u_r^\delta) - a u_r^\delta (u_r)_x, \quad 0<\alpha\le 1, \;\; r>0, \tag{3.1}$$

with the initial and boundary conditions:

$$u_{r+1}(x,0)=g(x), \;\; u_{r+1}(0,t)=y_0(t), \;\; u(1,t)=y_1(t), \;\; t\ge 0, \quad 0<x<1.$$

For applying the second kind Chebyshev wavelet method to Eq. (3.1), we suppose that

$$\frac{\partial^2 u_r}{\partial x^2} = \sum_{i=1}^{m'}\sum_{j=1}^{m'} c_{i,j}^{r+1}\psi_i(x)\psi_j(t) = \Psi^T(x)C^{r+1}\Psi(t), \tag{3.2}$$

by applying the integral operator I^2 with respect to x on Eq. (3.2)

$$u_{r+1}(x,t) = (P^2)^T C^{r+1}\Psi(t) + p(t)x + q(t), \tag{3.3}$$

by using the boundary conditions and putting $x=0$ and $x=1$, we get:

$$\begin{aligned} x=0: &\qquad q(t)=y_0(t),\\ x=1: &\qquad p(t)=y_1(t)-y_0(t)-(P^2(1))^T C^{r+1}\Psi(t). \end{aligned}$$

Now Eq. (3.3) can be rewritten as:

$$u_{r+1}(x,t) = (P^2)^T C^{r+1}\Psi(t) - x\left((P^2(1))^T C^{r+1}\Psi(t)\right) + x(y_1(t)-y_0(t)) + y_0(t). \tag{3.4}$$

For simplicity, we put the right-hand-side of Eq. (3.1) equal to $S(x,t)$:

$$S(x,t) = b u_r(1-u_r^\delta) - a u_r^\delta (u_r)_x = \sum_{i=1}^{m'}\sum_{j=1}^{m'} m_{i,j}\psi_i(x)\psi_j(t) = \Psi^T(x)M\Psi(t), \tag{3.5}$$

where $m_{i,j} = \langle \psi_i(x), \langle S(x,t), \psi_j(t)\rangle\rangle$. By substituting Eqs. (3.5) and (3.2) in Eq. (3.1) we have

$$\frac{\partial^\alpha u_{r+1}}{\partial t^\alpha} = \Psi^T(x)C^{r+1}\Psi(t) + \Psi^T(x)M\Psi(t). \tag{3.6}$$

By applying the fractional integral operator I^α with respect to t to Eq. (3.6) and using the initial condition, we obtain:

$$u_{r+1}(x,t) = \Psi^T(x)C^{r+1}P^\alpha + \Psi^T(x)MP^\alpha + g(x). \tag{3.7}$$

From Eqs. (3.7) and (3.4):

$$K(x,t) + (P^2)^T C^{r+1}\Psi(t) - x\left((P^2(1))^T C^{r+1}\Psi(t)\right) - \Psi^T(x)C^{r+1}P^\alpha - \Psi^T(x)MP^\alpha = 0, \tag{3.8}$$

where $K(x,t) = -g(x) + x\big(y_1(t)-y_0(t)\big) + y_0(t)$. By discretizing Eq. (3.8) and using collocation points, we have the following matrix form:

$$\left((P^2)^T - V^{2,1,x}\right)C^{r+1}\Psi - \Psi^T C^{r+1}P^\alpha - \Psi^T M P^\alpha + K = 0, \tag{3.9}$$

where Ψ is the $2^{k-1}M \times 2^{k-1}M$ second kind Chebyshev wavelet matrix, $V^{2,1,x} = xI^2(\Psi(1))^T = x(P^2(1))^T$ is the $2^{k-1}M \times 2^{k-1}M$ matrix defined in (2.11) and $P^\alpha = I^\alpha(\Psi)$ is $2^{k-1}M \times 2^{k-1}M$ matrix. Also, $K = K(x_i,t_j), \quad i,j = 1,2,\dots,2^{k-1}M$ is a matrix determined using the collocation points.

Eq. (3.9) can be written as:

$$\underbrace{(\Psi^T)^{-1}\left((P^2)^T - V^{2,1,x}\right)}_{A} C^{r+1} - C^{r+1}\underbrace{P^{\alpha}(\Psi^{-1})}_{-B} = \underbrace{(\Psi^T)^{-1}\left(\Psi^T M P^{\alpha} - K\right)(\Psi^{-1})}_{D}, \tag{3.10}$$

which is the Sylvester equation $AX + XB = D$. By solving Eq. (3.10) for C^{r+1}, which is $2^{k-1}M \times 2^{k-1}M$ coefficient matrix, and substituting C^{r+1} in Eqs. (3.7) or (3.4), we get the solution $u_{r+1}(x,t)$ at the collocation points. Suppose an initial approximation $u_0(x,t)$, we get a linear fractional partial differential equation in $u_1(x,t)$ by substituting $r = 0$ in Eq. (3.1), which is solved by the illustrated procedure. Similarly, for $r = 1$, we obtain $u_2(x,t)$, and so on.

3.1. **Numerical Examples.** To demonstrate the efficiency of the Chebyshev wavelet Picard method (CWPM) for the generalized Burger and Burgers-Fisher equations, we solve some numerical examples for different values of a, b, δ, and α. The proposed method is compared with the Haar wavelet Picard method (HWPM) [36] and HPM [33] to demonstrate its capability.

Example 3.1. We consider Eq. (1.1) with the following initial and boundary conditions:

$$u(x,0) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}x\right)\right)^{\frac{1}{\delta}},$$

$$u(0,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}\left[-\left(\frac{a^2+b(1+\delta)^2)}{a(1+\delta)}\right)t\right]\right)\right)^{\frac{1}{\delta}},$$

$$u(1,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{a\delta}{2(1+\delta)}\left[1-\left(\frac{a^2+b(1+\delta)^2)}{a(1+\delta)}\right)t\right]\right)\right)^{\frac{1}{\delta}},$$

which the exact solution for $\alpha = 1$ is Eq. (1.2).

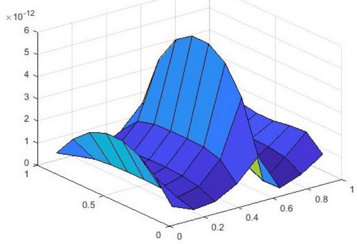
We suppose $u_0(x,t)$ as an initial approximation and apply the second Chebyshev wavelet Picard iteration technique for different values of a, b, α, and $\delta = 1$. Tables 1 and 2 show the approximate solutions obtained for different values α, which solutions at different values of α converge to the exact solution at $\alpha = 1$ when α approaches 1.

The results which are obtained by using the present method were compared with the homotopy perturbation method (HPM) [36] and the Haar wavelet Picard method (HWPM) [33], and are shown in Tables 3 and 4. Figures 1 and 2 show that numerical solutions are in very good coincidence with the exact solution, which happened by increasing the resolution M or iteration J.

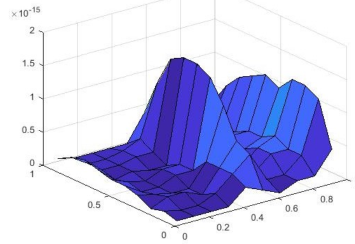
Table 1. Absolute error of approximate solutions obtained by using Chebyshev wavelet Picard with $\delta = 1, M = 8, K = 2$ for different values α for Example 3.1.

	$\delta=1$ $J=4$ $a=0.01$, $b=0.01$			$\delta=1$ $J=4$ $a=0.001$, $b=0.001$		
(x,t)	$\alpha=0.6$	$\alpha=0.9$	$\alpha=1$	$\alpha=0.6$	$\alpha=0.9$	$\alpha=1$
$(\frac{1}{32},\frac{1}{32})$	1.919×10^{-6}	5.028×10^{-7}	0	1.922×10^{-5}	5.0369×10^{-6}	1.110×10^{-16}
$(\frac{7}{32},\frac{7}{32})$	7.991×10^{-6}	2.513×10^{-6}	0	8.005×10^{-5}	2.5177×10^{-5}	0
$(\frac{13}{32},\frac{13}{32})$	6.939×10^{-6}	1.853×10^{-6}	0	6.953×10^{-5}	1.8573×10^{-5}	1.110×10^{-16}
$(\frac{19}{32},\frac{19}{32})$	3.396×10^{-6}	6.009×10^{-6}	0	3.404×10^{-5}	6.0227×10^{-6}	0
$(\frac{25}{32},\frac{25}{32})$	3.325×10^{-7}	2.489×10^{-7}	0	2.332×10^{-6}	2.4951×10^{-6}	1.110×10^{-16}
$(\frac{31}{32},\frac{31}{32})$	3.091×10^{-7}	1.418×10^{-7}	1.110×10^{-16}	3.099×10^{-6}	1.423×10^{-6}	0

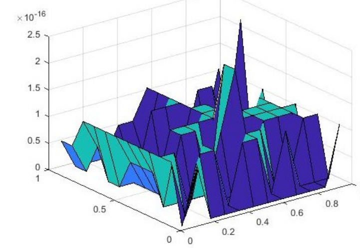
Example 3.2. For $b = 0$, Eq. (1.1) is reduced to the generalized Burger equation. We have taken different values of a, δ. Table 5 shows the approximate solution obtained for different values of α, where the solutions at different values of α converge to the exact solution at $\alpha = 1$, when α approaches 1.



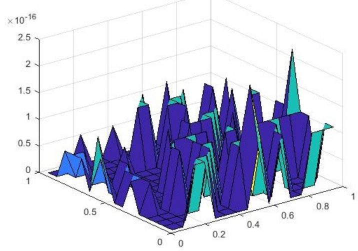
(a) $M = 3$, $a = b = 0.1$.



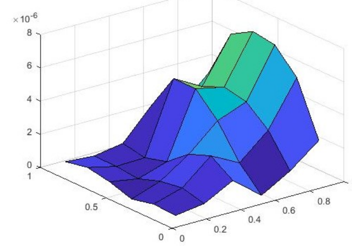
(b) $M = 4$, $a = b = 0.1$.



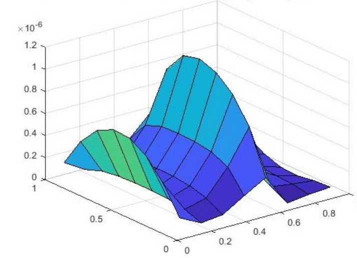
(c) $M = 6$, $a = b = 0.1$.



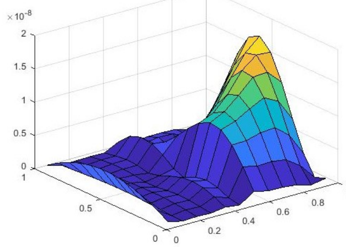
(d) $M = 8$, $a = b = 0.1$.



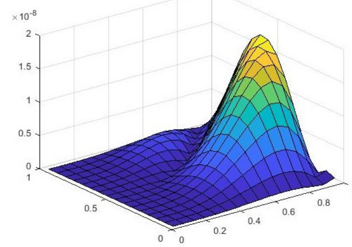
(e) $M = 8$, $a = b = 1$.



(f) $M = 4$, $a = b = 1$.



(g) $M = 6$, $a = b = 1$.

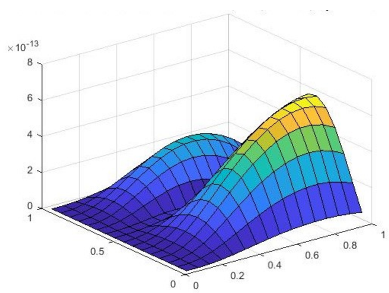


(h) $M = 8$, $a = b = 1$.

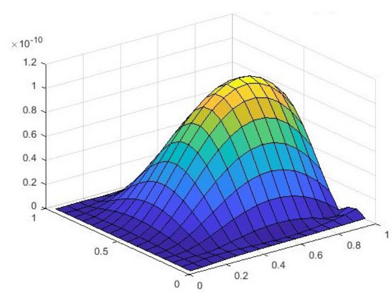
FIGURE 1. Absolute error for 6th iteration of CWPM solutions with $\alpha = \delta = 1, K = 2$ and different value of M, a, and b in Example 3.1.

TABLE 2. Absolute error of approximate solutions obtained by using Chebyshev wavelet Picard with $\delta = 1, M = 8, K = 2$ for different values α in Example 3.1.

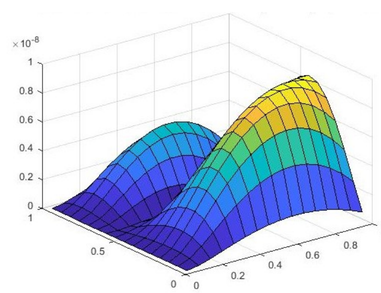
	$\delta = 1 \quad J = 4 \quad a = 0.5, \quad b = 0.5$			$\delta = 1 \quad J = 4 \quad a = 1, \quad b = 1$		
(x, t)	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 1$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 1$
$(\frac{1}{32}, \frac{1}{32})$	1.044×10^{-4}	2.758×10^{-4}	1.320×10^{-12}	2.247×10^{-3}	5.979×10^{-4}	7.507×10^{-11}
$(\frac{7}{32}, \frac{7}{32})$	4.387×10^{-3}	1.378×10^{-3}	1.185×10^{-9}	9.466×10^{-3}	2.976×10^{-3}	8.511×10^{-8}
$(\frac{13}{32}, \frac{13}{32})$	3.838×10^{-3}	1.025×10^{-3}	4.025×10^{-9}	8.102×10^{-3}	2.165×10^{-3}	3.466×10^{-7}
$(\frac{19}{32}, \frac{19}{32})$	1.848×10^{-3}	3.201×10^{-4}	1.458×10^{-9}	3.372×10^{-3}	5.038×10^{-4}	3.322×10^{-7}
$(\frac{25}{32}, \frac{25}{32})$	3.740×10^{-5}	1.703×10^{-5}	2.059×10^{-9}	8.206×10^{-4}	6.468×10^{-4}	2.254×10^{-7}
$(\frac{31}{32}, \frac{31}{32})$	2.076×10^{-4}	9.066×10^{-5}	3.098×10^{-10}	7.154×10^{-4}	2.773×10^{-4}	1.161×10^{-7}



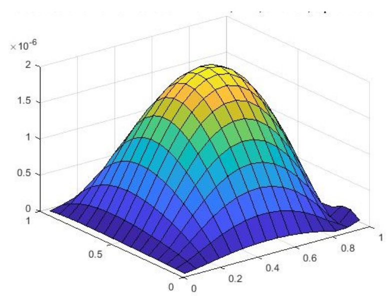
(a) 4th iteration, $a = b = 0.1$.



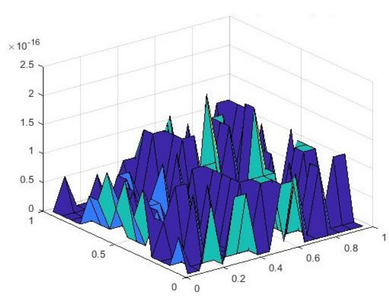
(b) 3th iteration, $a = b = 0.1$.



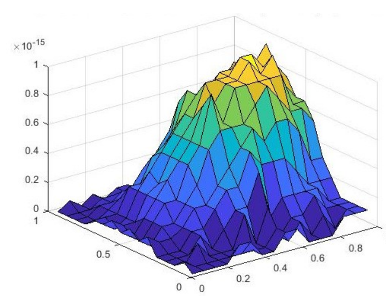
(c) 2th iteration, $a = b = 0.1$.



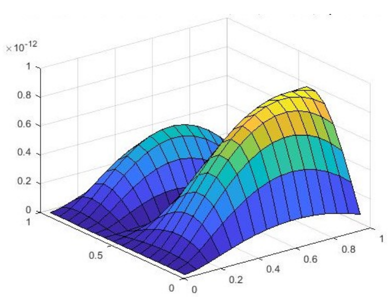
(d) 1th iteration, $a = b = 0.1$.



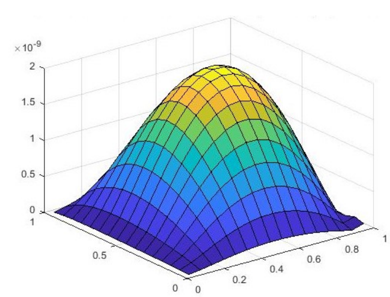
(e) 4th iteration, $a = b = 0.01$.



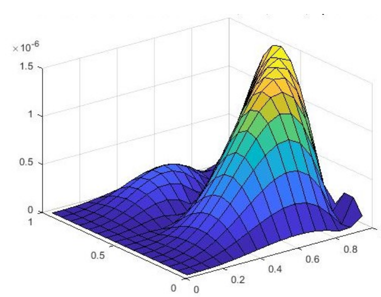
(f) 3th iteration, $a = b = 0.01$.



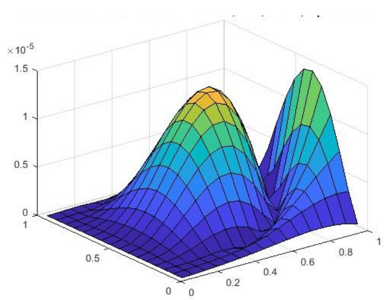
(g) 2th iteration, $a = b = 0.01$.



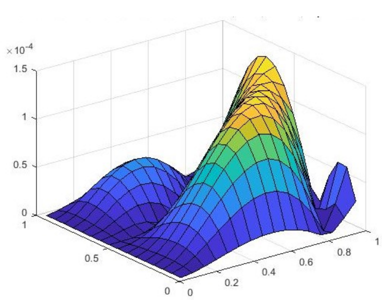
(h) 1th iteration, $a = b = 0.01$.



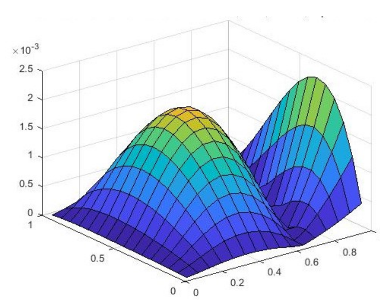
(i) 4th iteration, $a = b = 1$.



(j) 3th iteration, $a = b = 1$.



(k) 2th iteration, $a = b = 1$.



(l) 1th iteration, $a = b = 1$.

FIGURE 2. Absolute error for CWPM solutions for different iterations and different values of a, b with $\alpha = 1, K = 2, M = 8$, and $\delta = 1$ in Example 3.1.

Table 3. Comparison of approximate solutions by obtained using Chebyshev wavelet Picard method (CWPM) with Haar wavelet Picard method (HWPM) and homotopy perturbation method (HPM) for $M = 8, K = 2$ in Example 3.1.

	$\delta = 1 \quad J = 4 \quad a = b = 0.001, \quad \alpha = 1$			$\delta = 1 \quad J = 4 \quad a = b = 0.01, \quad \alpha = 1$		
(x,t)	CWPM	HWP[36]	HPM[33]	CWPM	HWPM[36]	HPM[33]
$(\frac{1}{32}, \frac{1}{32})$	0	6.661×10^{-16}	3.331×10^{-16}	1.110×10^{-16}	7.136×10^{-13}	3.179×10^{-13}
$(\frac{7}{32}, \frac{7}{32})$	0	1.776×10^{-15}	1.090×10^{-13}	0	1.800×10^{-12}	1.090×10^{-10}
$(\frac{13}{32}, \frac{13}{32})$	0	4.44×10^{-15}	6.983×10^{-13}	1.110×10^{-16}	4.531×10^{-12}	6.984×10^{-10}
$(\frac{19}{32}, \frac{19}{32})$	0	4.885×10^{-15}	2.180×10^{-12}	0	4.908×10^{-12}	2.180×10^{-9}
$(\frac{25}{32}, \frac{25}{32})$	0	3.553×10^{-15}	4.967×10^{-12}	1.110×10^{-16}	3.553×10^{-12}	4.967×10^{-9}
$(\frac{31}{32}, \frac{31}{32})$	8.882×10^{-16}	9.470×10^{-12}	3.098×10^{-10}	0	8.322×10^{-13}	9.470×10^{-9}

Table 4. Comparison of approximate solution obtained by using Chebyshev wavelet Picard method (CWPM) with Haar wavelet Picard method (HWPM) and homotopy perturbation method (HPM) for $M = 8, K = 2$ in Example 3.1.

	$\delta = 1 \quad J = 4 \quad a = b = 0.1, \quad \alpha = 1$			$\delta = 1 \quad J = 4 \quad a = b = 0.5, \quad \alpha = 1$		
(x,t)	CWPM	HWP[36]	HPM[33]	CWPM	HWPM[36]	HPM[33]
$(\frac{1}{32}, \frac{1}{32})$	1.110×10^{-16}	7.352×10^{-10}	5.408×10^{-10}	1.320×10^{-12}	1.035×10^{-7}	7.709×10^{-8}
$(\frac{7}{32}, \frac{7}{32})$	6.939×10^{-14}	1.961×10^{-9}	1.855×10^{-7}	1.185×10^{-9}	3.454×10^{-7}	2.641×10^{-5}
$(\frac{13}{32}, \frac{13}{32})$	2.161×10^{-13}	4.858×10^{-9}	1.188×10^{-6}	4.025×10^{-9}	8.077×10^{-7}	1.687×10^{-4}
$(\frac{19}{32}, \frac{19}{32})$	5.995×10^{-15}	5.264×10^{-9}	3.708×10^{-6}	1.458×10^{-9}	8.634×10^{-7}	5.243×10^{-4}
$(\frac{25}{32}, \frac{25}{32})$	2.391×10^{-13}	3.816×10^{-9}	8.446×10^{-6}	2.059×10^{-9}	6.102×10^{-7}	1.187×10^{-3}
$(\frac{31}{32}, \frac{31}{32})$	5.795×10^{-14}	8.886×10^{-10}	1.609×10^{-5}	3.099×10^{-10}	1.356×10^{-7}	2.245×10^{-3}

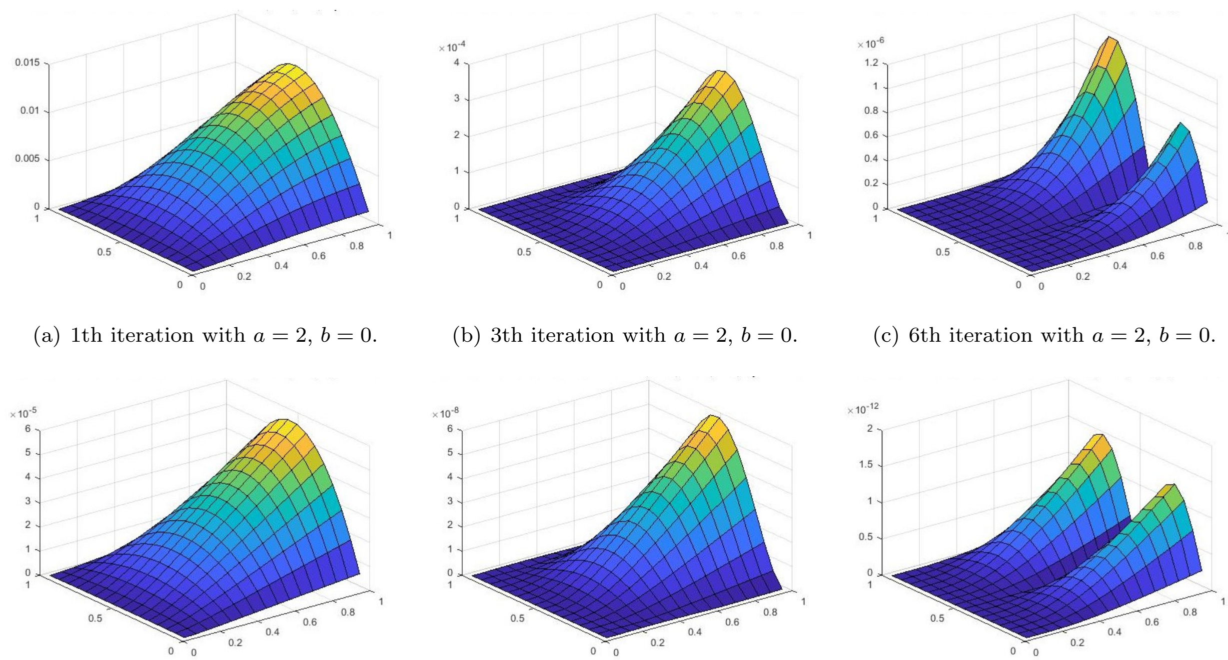
The results of the present method were compared with the homotopy perturbation method (HPM) [33] and the Haar wavelet Picard method (HWPM) [36] in Tables 6 and 7. Also, Figures 3 and 4 show that the numerical solutions coincide very well with the exact solution, which happens by increasing the resolution M or the iteration J.

Table 5. Absolute error of approximate solutions obtained by using Chebyshev wavelet Picard with $b = 0, M = 8, K = 2$ for different value α and δ in Example 3.2.

	$\delta = 1 \quad J = 4 \quad a = 0.5, \quad b = 0$			$\delta = 2 \quad J = 4 \quad a = 0.1, \quad b = 0$		
(x, t)	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 1$	$\alpha = 0.6$	$\alpha = 0.9$	$\alpha = 1$
$(\frac{1}{32}, \frac{1}{32})$	1.157×10^{-4}	3.064×10^{-5}	7.705×10^{-14}	5.521×10^{-6}	2.449×10^{-6}	1.377×10^{-6}
$(\frac{7}{32}, \frac{7}{32})$	4.875×10^{-4}	1.531×10^{-4}	6.272×10^{-11}	3.114×10^{-5}	3.269×10^{-5}	1.872×10^{-5}
$(\frac{13}{32}, \frac{13}{32})$	4.298×10^{-4}	1.147×10^{-4}	1.161×10^{-10}	3.823×10^{-5}	3.152×10^{-5}	2.935×10^{-5}
$(\frac{19}{32}, \frac{19}{32})$	2.142×10^{-4}	3.811×10^{-5}	3.444×10^{-10}	3.347×10^{-5}	3.029×10^{-5}	3.001×10^{-5}
$(\frac{25}{32}, \frac{25}{32})$	1.614×10^{-5}	1.538×10^{-5}	8.376×10^{-10}	2.069×10^{-5}	2.071×10^{-5}	2.146×10^{-5}
$(\frac{31}{32}, \frac{31}{32})$	1.959×10^{-5}	9.044×10^{-6}	2.276×10^{-10}	3.175×10^{-6}	3.558×10^{-6}	3.832×10^{-6}

4. Conclusion

In this study, we have successfully applied a combination of the operational matrix of the Chebyshev wavelet based on the collocation method and the Picard iteration technique to solve the time-fractional Burgers-Fisher equation. We have transformed the nonlinear fractional partial differential equation to a linear differential equation using the Picard iteration technique and then converted the obtained linear equation into a Sylvester equation employing the Chebyshev



(a) 1th iteration with $a = 2$, $b = 0$. (b) 3th iteration with $a = 2$, $b = 0$. (c) 6th iteration with $a = 2$, $b = 0$.

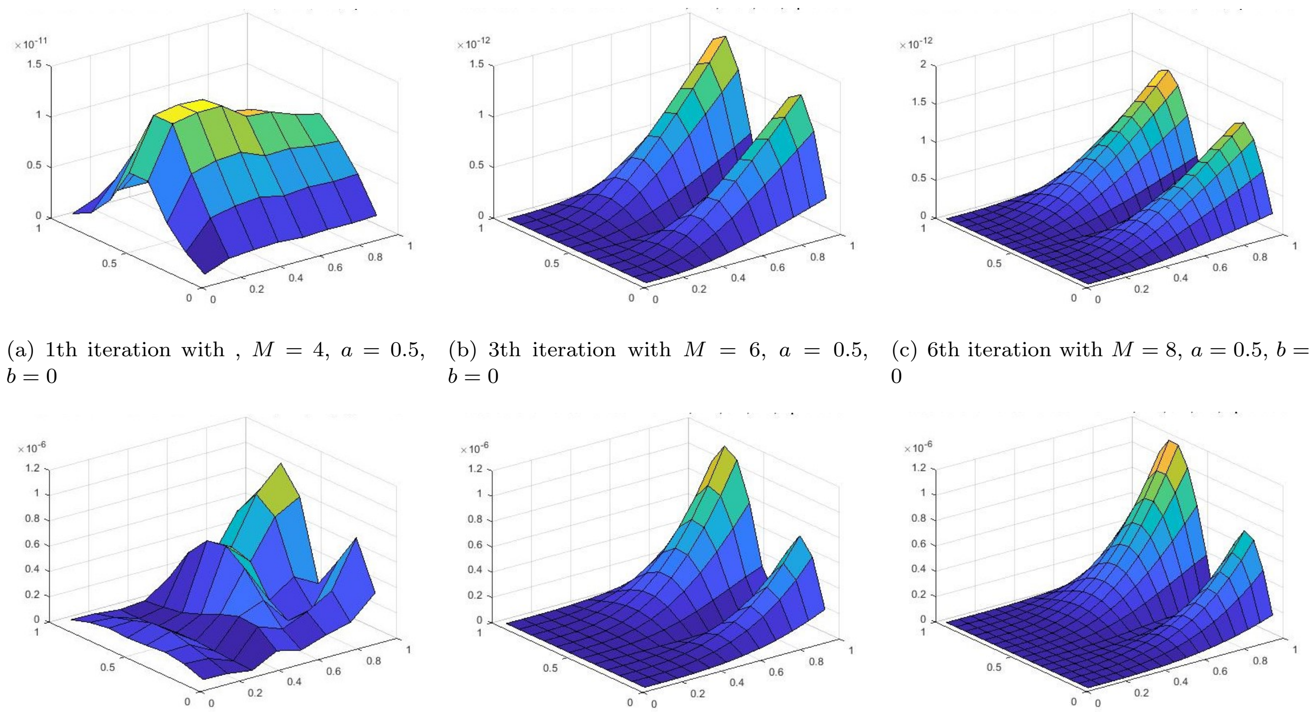
(d) 1th iteration with $a = 0.5$, $b = 0$. (e) 3th iteration with $a = 2$, $b = 0$. (f) 6th iteration with $a = 2$, $b = 0$.

Figure 3. Absolute error for CWPM solutions for different iterations and different values of a, b with $\alpha = 1, K = 2, M = 8$, and $\delta = 1$ in Example 3.2, which show that numerical solutions are in very good coincidence with the exact solutions which happened by increasing the iterations.

Table 6. Comparison of the approximate solution obtained by using the Chebyshev wavelet Picard method (CWPM) with the Haar wavelet Picard method (HWPM) and the Adomian decomposition method (HPM) for $M = 8, K = 2$ in Example 3.2.

	$\delta = 1 \quad J = 4 \quad a = 0.1, \ b = 0, \ \alpha = 1$			$\delta = 1 \quad J = 4 \quad a = 0.5, \ b = 0, \ \alpha = 1$		
(x,t)	CWPM	HWPM[36]	HPM[33]	CWPM	HWPM[36]	HPM[33]
$(\frac{1}{32}, \frac{1}{32})$	0	5.768×10^{-14}	5.861×10^{-13}	7.705×10^{-14}	1.325×10^{-9}	1.077×10^{-9}
$(\frac{7}{32}, \frac{7}{32})$	8.882×10^{-16}	8.424×10^{-12}	2.010×10^{-10}	6.272×10^{-11}	2.929×10^{-9}	5.852×10^{-7}
$(\frac{13}{32}, \frac{13}{32})$	1.887×10^{-15}	1.407×10^{-11}	1.288×10^{-9}	1.161×10^{-10}	7.681×10^{-9}	3.740×10^{-6}
$(\frac{19}{32}, \frac{19}{32})$	3.830×10^{-15}	1.440×10^{-11}	4.019×10^{-9}	3.445×10^{-10}	7.934×10^{-9}	1.164×10^{-5}
$(\frac{25}{32}, \frac{25}{32})$	1.066×10^{-14}	1.022×10^{-11}	9.154×10^{-9}	8.376×10^{-10}	5.212×10^{-9}	2.639×10^{-5}
$(\frac{31}{32}, \frac{31}{32})$	2.9976×10^{-15}	1.937×10^{-12}	1.745×10^{-8}	2.277×10^{-10}	1.255×10^{-9}	5.003×10^{-5}

wavelet collocation method. The results obtained have been compared with the exact solutions for HPM and HWPM. The results have shown that the numerical solutions are in very good coincidence with the exact solutions, which occurred by increasing the iterations or the level of resolution or both. The obtained results have demonstrated the proposed method's accuracy, efficiency, and reliability. The agreement between the present numerical results, which



(a) 1th iteration with , $M = 4$, $a = 0.5$, $b = 0$
(b) 3th iteration with $M = 6$, $a = 0.5$, $b = 0$
(c) 6th iteration with $M = 8$, $a = 0.5$, $b = 0$
(d) 1th iteration with $M = 4$, $a = 2$, $b = 0$
(e) 3th iteration with $M = 6$, $a = 2$, $b = 0$
(f) 6th iteration with $M = 8$, $a = 2$, $b = 0$

FIGURE 4. Absolute error for CWPM solutions with $\alpha = \delta = 1, K = 2$ and different values of M, a, and b in Example 3.2, which shows that numerical solutions are in very good coincidence with exact solutions which happened by increasing the resolution.

TABLE 7. Comparison of the approximate solutions obtained by using the Chebyshev wavelet Picard method (CWPM) with the Haar wavelet Picard method (HWPM) and homotopy perturbation method (HPM) for $M = 8, K = 2$ in Example 3.2.

	$\delta = 0.5 \quad J = 4 \quad a = 0.2, \ b = 0, \ \alpha = 1$			$\delta = 2 \quad J = 4 \quad a = 0.2, \ b = 0, \ \alpha = 1$		
(x,t)	CWPM	HWPM[36]	HPM[33]	CWPM	HWPM[36]	HPM[33]
$(\frac{1}{32}, \frac{1}{32})$	4.649×10^{-7}	4.153×10^{-7}	1.892×10^{-7}	5.517×10^{-6}	4.931×10^{-6}	6.578×10^{-8}
$(\frac{7}{32}, \frac{7}{32})$	6.226×10^{-5}	6.253×10^{-5}	9.519×10^{-6}	7.550×10^{-5}	7.583×10^{-5}	2.975×10^{-6}
$(\frac{13}{32}, \frac{13}{32})$	9.707×10^{-5}	9.722×10^{-5}	3.363×10^{-5}	1.191×10^{-4}	1.193×10^{-4}	9.366×10^{-6}
$(\frac{19}{32}, \frac{19}{32})$	9.876×10^{-5}	9.881×10^{-5}	7.344×10^{-5}	1.225×10^{-4}	1.225×10^{-4}	1.803×10^{-5}
$(\frac{25}{32}, \frac{25}{32})$	7.027×10^{-5}	7.278×10^{-5}	1.297×10^{-4}	8.809×10^{-5}	8.809×10^{-5}	2.770×10^{-5}
$(\frac{31}{32}, \frac{31}{32})$	1.249×10^{-5}	1.248×10^{-5}	2.031×10^{-4}	1.582×10^{-5}	1.581×10^{-5}	3.707×10^{-5}

are obtained by the Chebyshev wavelet Picard method, and the exact solutions appears to be very satisfactory, as illustrated in the presented Tables and Figures. However, the Chebyshev wavelet iteration method provides a more accurate and better solution compared to HPM and HWPM. The present scheme is straightforward, effective, and appropriate for obtaining numerical solutions for nonlinear fractional partial differential equations.

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