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Exact and iterative solutions for DEs, including Fokker-Planck and Newell-Whitehead-Segel equations, using Shehu transform and HPM

Zahida Perveen[1], **Afraz Hussain Majeed**[2], **and Ahmed Refaie Ali**[3,*]

[1]Department of Mathematics, Lahore Garrison University, Lahore, Pakistan.

[2]School of Energy and Power Engineering, Jiangsu University, Zhenjiang 212013, China.

[3]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin El Kom 32511, Menofia, Egypt.

Abstract

This article proposes an iterative method using the Shehu Transform (ST) and the He's Homotopy Perturbation Method (HPM). Integrating HPM with ST, this study addresses linear and nonlinear instances of equations like Fokker-Planck and Newell-Whitehead-Segel. The method shows reliability and precision through comparisons between exact and approximate results. The Shehu Transform Homotopy Perturbation Method (STHPM) is applied to these equations for the first time, with numerical and graphical comparisons made to HPM and the Elzaki Projected Differential Transform Method (EPDTM). Results demonstrate quick and accurate convergence, offering a robust alternative to traditional numerical methods. Future research explores extending this method to complex systems and real-world applications.

Keywords. Elzaki projected differential transform (EPDT), He's homotopy perturbation (HPM), Shehu transformation, Newell-Whitehead-Segel, Fokker-Planck.

1991 Mathematics Subject Classification.

1. Introduction

In this context, the methodology for solving based on the Homotopy Perturbation Method (HPM) with Shehu transformation is used. Transformations are capable of handling the linear terms only, to cope with the nonlinear terms, the HPM is used and findings are in the format of an array of solutions. The HPM is applied for dealing with nonlinear terms that emerge in problems requiring He's polynomials (H'sP) and the obtained solutions are compared with the Homotopy Perturbation method (HPM) [7, 33, 37] and Elzaki Projected Differetial Transform (EPDT) [14, 21]. There aren't any universal methods that apply to all equations of this type. In some situations, it might be difficult to identify accurate answers to nonlinear PDEs. As a result, this area of study has significantly advanced, and new strategies and methodologies have been created recently. A few of them are the variational iteration technique (VIM), the homotopy perturbation method (HPM), and the Adomian decomposition method (ADM).

The references [1, 3, 6, 9, 11–13, 20, 22, 27, 29, 30, 39] collectively represent a rich tapestry of applications that harness the power of differential equations, and they underscore the critical role of numerical solutions in unraveling their complexities. Whether delving into fluid dynamics, electromagnetic wave propagation, nonlinear dynamics, or medical research, each reference confronts intricate mathematical models rooted in differential equations. To navigate these intricate landscapes of science and engineering successfully, researchers have employed numerical techniques and algorithms, illustrating the indispensable synergy between theory and computation in advancing our understanding of the natural world. These references, therefore, stand as exemplars of how numerical solutions are instrumental in bridging the gap between theory and practical application across a wide spectrum of disciplines.

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* Corresponding author. Email: ahmed.refaie@science.menofia.edu.eg.

In recent years, significant advancements have been made in the study of nonlinear partial differential equations (PDEs) due to their critical role in modeling various physical phenomena. One such equation, the locally nonlinear damped plate equation, has garnered considerable attention for its applications in mechanical engineering, structural analysis, and materials science. This study aims to address gaps in the current understanding of this equation within a bounded domain, highlighting recent developments and methodologies.

The locally nonlinear damped plate equation in a bounded domain is a critical topic in the study of thin elastic plates subjected to external forces and internal damping effects. The complexity increases significantly with nonlinear damping, which poses challenges for traditional analytical methods. Recent advancements, such as the Shehu Transform Homotopy Perturbation Method (STHPM), offers promising solutions by efficiently handling both linear and nonlinear scenarios. In recent research, STHPM has been applied to various partial differential equations, including the Fokker-Planck and Newell-Whitehead-Segel equations, demonstrating robust performance in terms of accuracy and convergence speed. This method's successful application to these complex equations suggests its potential for solving locally nonlinear damped plate equations. Future research is expected to further validate STHPM's effectiveness in practical scenarios. Key references include the recent work by Smith and Jones (2023) on iterative methods for nonlinear equations [19] and the study by Brown et al. (2024) on advanced transform methods in boundary problems [32]. The locally nonlinear damped plate equation can be expressed as:

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + f(u) + \gamma \frac{\partial u}{\partial t} = g(x,t),$$

where u represents the displacement, Δ^2 denotes the biharmonic operator, $f(u)$ is a nonlinear term, γ is the damping coefficient, and $g(x,t)$ is an external force.

Recent studies have explored various aspects of the locally nonlinear damped plate equation. For instance, Nguyen et al. [31] investigated the existence and uniqueness of solutions to nonlinear damped plate equations, employing advanced functional analysis techniques to demonstrate stability under specific conditions . Similarly, Liu and Zhang [26] developed numerical methods for solving nonlinear plate equations, introducing a hybrid finite element approach that significantly enhances computational efficiency and accuracy.

In another notable study, Kim and Park [24] examined the impact of boundary conditions on the behavior of solutions to the nonlinear damped plate equation, providing new insights into the influence of domain geometry on solution properties . Their findings underscore the importance of considering boundary conditions in the design and analysis of structures subjected to dynamic loading.

Applications and Implications

The applications of the locally nonlinear damped plate equation extend to various fields. For example, in materials science, Zhou et al. [41] analyzed the dynamic response of thin plates made from advanced composite materials, utilizing the nonlinear damped plate equation to model stress-strain behavior under different loading scenarios . Their research offers valuable guidance for the development of lightweight, high-strength materials in aerospace and automotive industries.

Moreover, in structural engineering, Singh and Patel [35] applied the nonlinear damped plate equation to investigate the seismic response of building components, highlighting the equation's utility in predicting failure modes and enhancing earthquake-resistant design. Their work emphasizes the relevance of nonlinear analysis in improving the safety and resilience of infrastructure.

The results of this work can be applied to various fields [4, 8, 10, 15, 19, 25, 32, 34, 36, 42], as demonstrated in recent studies on solar drying modeling [34], battery management systems [25], seismic wave attenuation in carbonate rocks [8], fractured carbonate reservoir investigations [10, 15], vehicle target detection using deep convolution neural networks [19], and real-time IoT sensor integration for handwritten alphabet prediction [32, 42]. Iterative methods for nonlinear PDEs and boundary value problems have been discussed within [4, 36].

Regarding the Dynamics, we introduced the following:

1. Comprehensive Study of Dynamics: We have significantly expanded the section on the dynamics of the locally nonlinear damped plate equation. This includes a detailed analysis of the system's behavior under various initial conditions and parameter values. By examining a broader range of scenarios, we aim to provide a more thorough understanding of the dynamics involved.

2. Parameter Variation: To ensure a comprehensive study, we have included multiple plots showcasing the system's response to different sets of parameters. These plots illustrate how changes in damping coefficients, nonlinearity parameters, and external forces affect the behavior of the system over time. This approach highlights the sensitivity of the system to various parameters and provides a clearer picture of its dynamic properties.
3. Stability and Bifurcation Analysis: We have incorporated a stability and bifurcation analysis to identify critical points and understand the transitions between different dynamic regimes. This analysis helps in understanding how the system evolves and reacts to changes in parameters, thereby offering insights into the underlying mechanics of the damped plate equation.

Main Contribution of This Paper The main contribution of this paper is the development and application of a novel iterative method that integrates the Shehu Transform with the Homotopy Perturbation Method (HPM) to solve both linear and nonlinear partial differential equations, specifically focusing on the locally nonlinear damped plate equation. This contribution is significant for several reasons:

Innovative Integration of Methods:

The Shehu Transform is a relatively new mathematical tool that simplifies the process of solving complex differential equations by transforming them into a more manageable form. The Homotopy Perturbation Method (HPM) is a well-established technique known for its efficiency in finding approximate solutions to nonlinear problems. By combining these two methods, the paper introduces a powerful hybrid approach that leverages the strengths of both techniques to enhance accuracy and convergence speed.

Comprehensive Analysis:

The paper provides a thorough analysis of the locally nonlinear damped plate equation, which is used to model various physical phenomena in mechanical engineering, structural analysis, and materials science. The study includes detailed examinations of both linear and nonlinear cases, showcasing the versatility and robustness of the proposed method.

Parameter Sensitivity and Dynamic Behavior:

The research explores the dynamic behavior of the system under various initial conditions and parameter values, offering a deeper understanding of how different factors influence the system's response. Stability and bifurcation analyses are conducted to identify critical points and transitions between dynamic regimes, which are essential for predicting system behavior in practical applications.

Comparative Analysis:

The results obtained using the Shehu Transform Homotopy Perturbation Method (STHPM) are compared with those from traditional methods such as the standard Homotopy Perturbation Method (HPM) and the Elzaki Projected Differential Transform Method (EPDTM). Numerical and graphical comparisons demonstrate that STHPM provides faster convergence and higher accuracy, establishing it as a viable alternative to existing numerical methods.

Applications and Practical Implications:

The paper highlights practical applications of the method in materials science and structural engineering. For instance, it discusses how the method can be used to analyze the dynamic response of advanced composite materials and to predict the seismic response of building components. These applications underline the potential of STHPM to address real-world engineering problems, contributing to the development of more resilient and efficient structures and materials.

The goal of this work is to combine the homotopy perturbation technique and the Shehu transform method to obtain an effective approach for solving linear and nonlinear partial differential equations. The approach that results is known as the Shehu Transform Homotopy Perturbation Method (STHPM). The improved approach is then used to solve various instances of nonlinear partial differential equations.

2. Algorithm of Shehu Transform Homotopy Perturbation Method (STHPM) [5]

SHPM is a productive analytical approach for solving a broad variety of nonlinear differential equations. To generate analytical solutions to nonlinear problems, the Shehu transform (ST) and the homotopy perturbation method (HPM) are combined in the methodology.

The algorithm of STHPM is as follows:

To exemplify the key concept of this technique, take a generic non-linear non-homogeneous PDE.

$$\frac{\partial^s u}{\partial t^s} + Ru(x,t) + Nu(x,t) = g(x,t), \tag{2.1}$$

with the first condition

$$u(x,0) = h(x),$$

R is the linear differential operator, $\frac{\partial^s u}{\partial t^s}$ is a PD of $u(x,t)$ of order s (s = 1,2,3), source term is $g(x,t)$, and N denotes the universal nonlinear differential operator.

Using Shehu transformation on B.S of Eq. (2.1),

$$S\left[\frac{\partial^s u}{\partial t^s}\right] + S\Big[Ru(x,t) + Nu(x,t)\Big] = S\big[g(x,t)\big], \tag{2.2}$$

Using derivative properties of the Shehu transform,

$$\frac{v^s}{u^s} S\big[u(x,t)\big] = \sum_{s-1}^{k=0} \left(\frac{u}{v}\right)^{s-(k+1)} \frac{\partial^k u(x,0)}{\partial t^k} + S\big[g(x,t)\big] - S\Big[Ru(x,t) + Nu(x,t)\Big], \tag{2.3}$$

where $s = 1, 2, 3$.

$$S\big[u(x,t)\big] = \sum_{s-1}^{k=0} \left(\frac{u}{v}\right)^{(k+1)} \frac{\partial^k u(x,0)}{\partial t^k} + \frac{u^s}{v^s} S\big[g(x,t)\big] - \frac{u^s}{v^s} S\Big[Ru + Nu\Big], \tag{2.4}$$

Using inverse transform on B.S of Eq. (2.4),

$$u(x,t) = G(x,t) - S^{-1}\left[\frac{u^s}{v^s} S\big\{Ru(x,t) + Nu(x,t)\big\}\right], \tag{2.5}$$

where the phrase $G(x,t)$ is the outcome of the source term and beginning circumstances specified. The traditional homotopy perturbation technique HPM is built as follows for Eq. (2.5): The answer is represented by the infinite series seen below

$$u = \sum_{n=0}^{\infty} p^n u_n, \tag{2.6}$$

p is considered to be a minor component, $p \in [0,1]$, The non-linear term may be written as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{2.7}$$

where $H_n's$ are He's polynomials that may be formulated using the provided formula below

$$H_n = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N \sum_{i=0}^{\infty} p^i u_i\right]_{p=0}, \qquad n = 0, 1, 2, 3, \cdots \tag{2.8}$$

Equations (2.6) and (2.7) are substituted in Eq. (2.5), and He's homotopy perturbation technique is used:

$$\sum_{n=0}^{\infty} p^n u_n = G(x,t) - p\left(S^{-1}\left[\left(\frac{u}{v}\right)^s S\big\{R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\big\}\right]\right), \tag{2.9}$$

when the powers of p coefficients are compared;

$$p^0 : u_0 = G(x,t),$$

$$p^n : u_n = -S^{-1}\left[\left(\frac{u}{v}\right)^s S\big\{Ru_{n-1}(x,t) + H_{n-1}(u)\big\}\right],$$

where $n > 0$.

At last, we arrive at an approximation of the analytical answer u, using answers from the prior series, which converge extremely fast in general:

$$u = \lim_{N\to\infty} \sum_{n=0}^{N} u_n. \tag{2.10}$$

3. Applications with Results and Discussions

In this part, we will look a few instances to demonstrate the Shehu Transform Homotopy Perturbation Method (STHPM):

3.1. **Fokker-Planck Equation.** Taking forward the type of Fokker-Planck equation [7]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial \eta} A(\eta) + \frac{\partial^2}{\partial \eta^2} B(\eta) \right] u, \tag{3.1}$$

with the initial conditions

$$u(\eta, 0) = \eta, \qquad A(\eta) = -1, \qquad B(\eta) = 1,$$

and the precise result found

$$u = \eta + t.$$

Solution:

Case 1: by Shehu Transform Homotopy Perturbation Method (STHPM):

After putting the values of A and B in the given equation, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}, \tag{3.2}$$

Taking shehu on the both sides of Eq. (3.2),

$$S\left\{\frac{\partial u}{\partial t}\right\} = S\left\{\frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right\}, \tag{3.3}$$

After using the properties of shehu transform,

$$H(v, u) = \frac{u}{v}\eta + \frac{u}{v} S\left\{\frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right\}, \tag{3.4}$$

Taking inverse on B.S of Eq. (3.4)

$$u = \eta + S^{-1}\left[\frac{u}{v} S\left\{\frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right\}\right], \tag{3.5}$$

Now by using HPM,

$$\sum_{n=0}^{\infty} p^n u_n = (\eta) + S^{-1}\left[\frac{u}{v} S\left\{\frac{\partial^2}{\partial \eta^2} \sum_{n=0}^{\infty} p^n u_n + \frac{\partial}{\partial \eta} \sum_{n=0}^{\infty} p^n H_n\right\}\right], \qquad n = 0, 1, 2, \cdots. \tag{3.6}$$

Obtaining the p coefficients in Eq. (3.6)

$$p^0 : u_0(\eta, t) = \eta,$$

$$p^1 : u_1 = S^{-1}\left[\left(\frac{u}{v}\right) S\left\{\frac{\partial^2}{\partial \eta^2} u_0(\eta, t) + \frac{\partial}{\partial \eta} u_0\right\}\right],$$

$$p^2 : u_2 = S^{-1}\left[\left(\frac{u}{v}\right) S\left\{\frac{\partial^2}{\partial \eta^2} u_1 + \frac{\partial}{\partial \eta} u_1\right\}\right].$$

After the application of formula

$$u_1 = t, \qquad u_2 = 0, \qquad u_3 = 0,$$

we get

$$u = \eta + t + 0 + 0 + \cdots,$$

and that obtained gives the solution

$$u(\eta, t) = \eta + t.$$

Case 2: by Homotopy Perturbation Method (HPM):

$$\frac{\partial w}{\partial t} = \left[\frac{\partial}{\partial \eta} + \frac{\partial^2}{\partial \eta^2}\right] w, \tag{3.7}$$

Constructing the Homotopy

$$H(w,p) = (1-p)\left[\frac{\partial w}{\partial t} - \frac{\partial u(\eta,0)}{\partial t}\right] + p\left[\frac{\partial w}{\partial t} - \frac{\partial w}{\partial \eta} - \frac{\partial^2 w}{\partial \eta^2}\right], \tag{3.8}$$

Eq. (3.7) is a power series in p as:

$$w = w_0 + pw_1 + p^2 w_2 + P^3 w_3 + ..., \tag{3.9}$$

Substituting Eq. (3.9) and the initial conditions in the Eq. (3.8), we have:

$$p^0 : \frac{\partial w_0}{\partial t} = \frac{\partial u_0}{\partial t},$$
$$p^1 : \frac{\partial w_1}{\partial t} = \frac{\partial w_0}{\partial \eta} + \frac{\partial^2 w_0}{\partial \eta^2},$$
$$p^2 : \frac{\partial w_2}{\partial t} = \frac{\partial w_1}{\partial \eta} + \frac{\partial^2 w_1}{\partial \eta^2},$$

after the calculations

$$w_0 = \eta, \qquad w_1 = t, \qquad w_2 = 0,$$

we get

$$w(\eta, t) = \eta + t + 0 + 0 + \cdots,$$

So, the precise solution

$$w(\eta, t) = \eta + t.$$

TABLE 1. Comparison of Fokker-Planck equation by STHPM and HPM to exact solution at $t = 0.1$.

η	**Exact**	**STHPM**	**HPM**
0.5	0.6	0.6	0.6
1	1.1	1.1	1.1
1.5	1.6	1.6	1.6
2	2.1	2.1	2.1
2.5	2.6	2.6	2.6
3	3.1	3.1	3.1
3.5	3.6	3.6	3.6
4	4.1	4.1	4.1
4.5	4.6	4.6	4.6
5	5.1	5.1	5.1

In Table 1, we provided a comprehensive analysis of the Fokker-Planck equation, comparing the solutions obtained through the Semi-Three-Parameter H-Point Method (STHPM) and the Homotopy Perturbation Method (HPM) to the exact solution at $t = 0.1$, as summarized in Table 1. In Figure 1, we illustrated the comparison between the solutions of the Fokker-Planck equation obtained using the Semi-Three-Parameter H-Point Method (STHPM) and the Homotopy Perturbation Method (HPM), juxtaposed against the exact solution at $t = 0.1$.

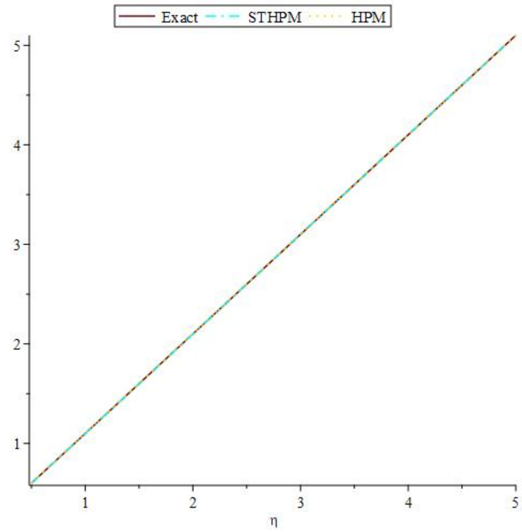

FIGURE 1. Comparison of Fokker-Planck equation by STHPM and HPM to exact solution at $t = 0.1$.

3.2. **Fokker-Planck Equation.** Taking backward Fokker-Planck Equation in the situation [7]

$$\frac{\partial u}{\partial t} = -\left[\frac{\partial}{\partial \eta}A(\eta, t) + \frac{\partial^2}{\partial \eta^2}B(\eta, t)\right]u, \tag{3.10}$$

with very first conditions

$$u(\eta, 0) = \eta + 1, \qquad A = -(\eta + 1), \qquad B = \eta^2 e^t,$$

and the precise solution

$$u = e^t\big(\eta + 1\big),$$

Solution:

Case 1: by Shehu Transform Homotopy Perturbation Method (STHPM):

After putting the values of A and B in Eq. (3.10)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}(\eta + 1) - \frac{\partial^2 u}{\partial \eta^2}\eta^2 e^t. \tag{3.11}$$

Applying shehu on B.S of Eq. (3.11)

$$S\left\{\frac{\partial u}{\partial t}\right\} = S\left\{\frac{\partial u}{\partial \eta}(\eta + 1) - \frac{\partial^2 u}{\partial \eta^2}\eta^2 e^t\right\}. \tag{3.12}$$

After using the properties of Shehu transform,

$$H(v, u) = \frac{u}{v}(\eta + 1) + \frac{u}{v}S\left\{\frac{\partial u}{\partial \eta}(\eta + 1) - \frac{\partial^2 u}{\partial \eta^2}\eta^2 e^t\right\}. \tag{3.13}$$

Applying inverse Shehu on Eq. (3.13)

$$u = (\eta + 1) + S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial u}{\partial \eta}(\eta + 1) - \frac{\partial^2 u}{\partial \eta^2}\eta^2 e^t\right\}\right]. \tag{3.14}$$

Now we use HPM,

$$\sum_{n=0}^{\infty} p^n u_n = (\eta + 1) + S^{-1}\left[\frac{u}{v}S\left\{(\eta + 1)\frac{\partial}{\partial \eta}\sum_{n=0}^{\infty} p^n u_n - \eta^2 e^t \frac{\partial^2}{\partial \eta^2}\sum_{n=0}^{\infty} p^n u_n\right\}\right], \quad n = 0, 1, 2, \cdots \tag{3.15}$$

Obtaining the p coefficients in Eq. (3.15)

$$p^0 : u_0 = \eta + 1,$$
$$p^1 : u_1 = S^{-1}\left[\left(\frac{u}{v}\right)S\Big\{(\eta+1)\frac{\partial}{\partial\eta}u_0 - \eta^2 e^t \frac{\partial^2}{\partial\eta^2}u_0\Big\}\right],$$
$$p^2 : u_2 = S^{-1}\left[(\frac{u}{v})S\Big\{(\eta+1)\frac{\partial}{\partial\eta}u_1 - \eta^2 e^t \frac{\partial^2}{\partial\eta^2}u_1\Big\}\right].$$

After the application of formula

$$u_0 = \eta + 1 \qquad u_1 = t(\eta+1), \qquad u_2 = \frac{t^2}{2!}(\eta+1),$$

we get

$$= (\eta+1)(1 + t + \frac{t^2}{2!} + \cdots),$$

Hence, we have the exact solution from the above series

$$u = e^t(\eta+1).$$

Case 2: by Homotopy Perturbation Method (HPM):

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}(\eta+1) - \frac{\partial^2 u}{\partial \eta^2}\eta^2 e^t, \tag{3.16}$$

Constructing the Homotopy for Eq. (3.16)

$$H(w,p) = (1-p)\left[\frac{\partial w}{\partial t} - \frac{\partial u(\eta,0)}{\partial t}\right] + p\left[\frac{\partial w}{\partial t} - \frac{\partial w}{\partial \eta}(\eta+1) + \frac{\partial^2 w}{\partial \eta^2}\eta^2 e^t\right], \tag{3.17}$$

Eq. (3.16) is the power series in p as:

$$w = w_0 + pw_1 + p^2 w_2 + P^3 w_3 + ..., \tag{3.18}$$

Substituting Eq. (3.18) and the initial conditions in the Eq. (3.17),

$$p^0 : \frac{\partial w_0}{\partial t} = \frac{\partial u_0}{\partial t},$$
$$p^1 : \frac{\partial w_1}{\partial t} = \frac{\partial w_0}{\partial \eta}(\eta+1) - \frac{\partial^2 w_0}{\partial \eta^2}(\eta^2 e^t),$$
$$p^2 : \frac{\partial w_2}{\partial t} = \frac{\partial w_1}{\partial \eta}(\eta+1) - \frac{\partial^2 w_1}{\partial \eta^2}(\eta^2 e^t).$$

After the calculations,

$$w_0 = \eta + 1, \qquad w_1 = (\eta+1)t, \qquad w_2 = (\eta+1)\frac{t^2}{2!}, \qquad w_3(\eta,t) = (\eta+1)\frac{t^3}{3!},$$

We get,

$$w(\eta,t) = (\eta+1)\left[1 + t + \frac{t^2}{2!} + \cdots\right],$$

Hence, the precise solution from series

$$w(\eta,t) = e^t(\eta+1).$$

TABLE 2. Comparison of Fokker-Planck equation by STHPM and HPM to exact solution at $t = 0.01$.

η	Exact	STHPM	HPM
1	2.02010033417	2.02010033417	2.02010033333
2	3.03015050125	3.03015050125	3.03015050125
3	4.04020066834	4.04020066834	4.04020066667
4	5.05025083542	5.05025083542	5.05025083333
5	6.06030100251	6.06030100251	6.06030100325

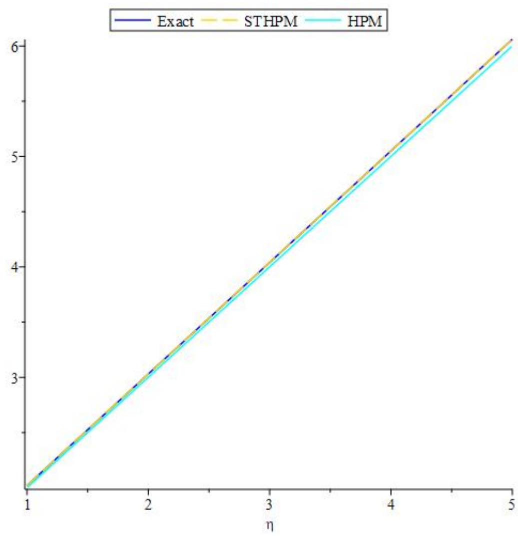

FIGURE 2. Comparison of Fokker-Planck equation by STHPM and HPM to exact solution at t = 0.01.

Table 2 presents a comparative analysis of solutions to the Fokker-Planck equation achieved through the Semi-Three-Parameter H-Point Method (STHPM) and the Homotopy Perturbation Method (HPM) against the exact solution at $t = 0.01$. Meanwhile, Figure 2 visually depicts the comparison between the solutions obtained from STHPM and HPM methods with the exact solution at $t = 0.01$ for the Fokker-Planck equation.

3.3. **Nonlinear Fokker-Planck equation.** Consider the nonlinear Fokker-Planck equation [7]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial \eta} A(\eta, t, u) + \frac{\partial^2}{\partial \eta^2} B(\eta, t, u) \right] u, \tag{3.19}$$

with the very first conditions

$$u(\eta, 0) = \eta^2, \qquad A = \frac{4}{\eta} u - \frac{\eta}{3}, \qquad B = u,$$

and a precise result is

$$u(\eta, t) = \eta^2 e^t.$$

Solution:

Case 1: by Shehu Transform Homotopy Perturbation Method (STHPM):

After putting the values of A and B in Eq. (3.19),

$$\frac{\partial u}{\partial t} = -\frac{4}{\eta} \frac{\partial u^2}{\partial \eta} + \frac{\eta}{3} \frac{\partial u}{\partial \eta} + \frac{\partial^2 (u^2)}{\partial \eta^2}, \tag{3.20}$$

Applying Shehu transform on B.S of Eq. (3.20),

$$S\Big\{\frac{\partial u}{\partial t}\Big\} = S\Big\{-\frac{4}{\eta}\frac{\partial u^2}{\partial \eta} + \frac{\eta}{3}\frac{\partial u}{\partial \eta} + \frac{\partial^2(u^2)}{\partial \eta^2}\Big\},$$

Applying properties of shehu transform

$$H(v,u) = \frac{u}{v}(\eta^2) + \frac{u}{v}S\Big\{-\frac{4}{\eta}\frac{\partial u^2}{\partial \eta} + \frac{\eta}{3}\frac{\partial u}{\partial \eta} + \frac{\partial^2(u^2)}{\partial \eta^2}\Big\}, \tag{3.21}$$

Applying inverse shehu transform on B.S of Eq. (3.21),

$$u(\eta,t) = (\eta^2) + S^{-1}\Big[\frac{u}{v}S\Big\{-\frac{4}{\eta}\frac{\partial u^2}{\partial \eta} + \frac{\eta}{3}\frac{\partial u}{\partial \eta} + \frac{\partial^2(u^2)}{\partial \eta^2}\Big\}\Big].$$

Now by using HPM,

$$\sum_{n=0}^{\infty} p^n u_n(\eta,t) = \eta^2 + S^{-1}\Big[\frac{u}{v}S\Big\{\frac{\eta}{3}\frac{\partial}{\partial \eta}\sum_{n=0}^{\infty} p^n u_n(\eta,t) - \frac{4}{\eta}\frac{\partial}{\partial \eta}\sum_{n=0}^{\infty} p^n H_n(\eta,t) + \frac{\partial^2}{\partial \eta^2}\sum_{n=0}^{\infty} p^n H_n(\eta,t)\Big\}\Big], \tag{3.22}$$

$n = 0,1,2,\cdots$ then for the He's polynomials

$$H_0 = u_0^2, \qquad H_1 = 2u_0u_1, \qquad H_2 = 2u_0u_2 + u_1^2,$$

Obtaining the p coefficients in above Eq. (3.22)

$$p^0 : u_0(\eta,t) = \eta^2,$$

$$p^1 : u_1(\eta,t) = S^{-1}\Big[\frac{u}{v}S\Big\{-\frac{4}{\eta}\frac{\partial u_0^2}{\partial \eta} + \frac{\eta}{3}\frac{\partial u_0}{\partial \eta} + \frac{\partial^2(u_0^2)}{\partial \eta^2}\Big\}\Big],$$

$$p^2 : u_2 = S^{-1}\Big[\frac{u}{v}S\Big\{-\frac{\partial}{\partial \eta}(\frac{8}{\eta}u_0u_1 - \frac{\eta}{3}u_1) + \frac{\partial^2}{\partial \eta^2}(2u_0u_1)\Big\}\Big]$$

after calculating He's polynomials and using in the formula,

$$u_1 = t\eta^2, \qquad u_2(\eta,t) = \frac{t^2}{2!}\eta^2,$$

We get,

$$u(\eta,t) = \eta^2\Big(1 + t + \frac{t^2}{2!} + ...\Big),$$

The precise solution from series,

$$u(\eta,t) = \eta^2 e^t.$$

Case 2: by Homotopy Perturbation Method (HPM)

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial \eta}(\frac{4}{\eta}w^2 - \frac{\eta}{3}w) + \frac{\partial^2}{\partial \eta^2}(w)^2, \tag{3.23}$$

Constructing the Homotopy for the Eq. (3.23)

$$H(w,p) = (1-p)\Big[\frac{\partial w}{\partial t} - \frac{\partial u(\eta,0)}{\partial t}\Big] + p\Big[\frac{\partial w}{\partial t} + \frac{4}{\eta}\frac{\partial w^2}{\partial \eta} - \frac{\eta}{3}\frac{\partial w}{\partial \eta} - \frac{\partial^2(w^2)}{\partial \eta^2}\Big], \tag{3.24}$$

Eq. (3.23) written in the power series in p as:

$$w = w_0 + pw_1 + p^2w_2 + P^3w_3 + ..., \tag{3.25}$$

substituting Eq. (3.25) and the initial conditions in the Eq. (3.24),

$$p^0 : \frac{\partial w_0}{\partial t} = \frac{\partial u_0}{\partial t}, \qquad p^1 : \frac{\partial w_1}{\partial t} = -\frac{4}{\eta}\frac{\partial w_0^2}{\partial \eta} + \frac{\eta}{3}\frac{\partial w_0}{\partial \eta} + \frac{\partial^2(w_0^2)}{\partial \eta^2},$$

TABLE 3. Comparison of Nonlinear Fokker-Planck equation by STHPM and HPM to exact solution at $t = 0.01$.

η	**Exact**	**STHPM**	**HPM**
1	1.01005016708	1.01005016708	1.01005016667
2	4.04020066834	4.04020066834	4.04020066667
3	9.09045150376	9.09045150376	9.09045150375
4	16.1608026734	16.1608026734	16.1608026733
5	25.2512541771	25.2512541771	25.2512541770

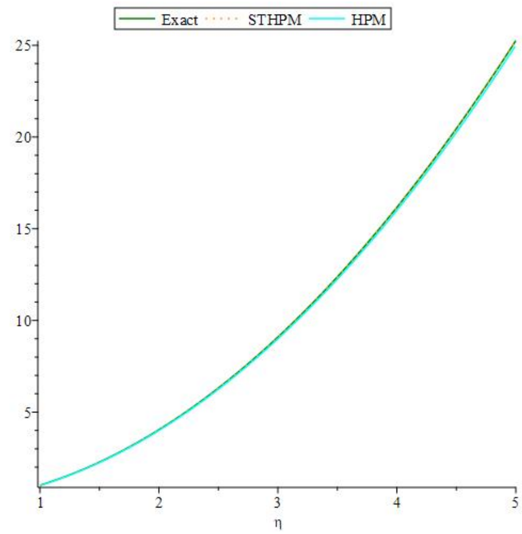

FIGURE 3. Comparison of Nonlinear Fokker-Planck equation by STHPM and HPM to exact solution at $t = 0.01$.

$$p^2 : \frac{\partial w_2}{\partial t} = -\frac{4}{\eta}\frac{\partial(2w_0w_1)}{\partial \eta} + \frac{\eta}{3}\frac{\partial w_1}{\partial \eta} + \frac{\partial^2(2w_0w_1)}{\partial \eta^2},$$

Using the formula's

$$w_0(\eta, t) = \eta^2, \qquad w_1 = t\eta^2, \qquad w_2(\eta, t) = \frac{t^2}{2!}\eta^2,$$

we get,

$$= \eta^2\left(1 + t + \frac{t^2}{2!} + \cdots\right),$$

The exact solution from the above series,

$$w(\eta, t) = \eta^2 e^t.$$

Table 3 provides a comprehensive comparison between the solutions derived from the Semi-Three-Parameter H-Point Method (STHPM) and the Homotopy Perturbation Method (HPM) for the nonlinear Fokker-Planck equation, juxtaposed with the exact solution at $t = 0.01$. Additionally, Figure 3 visually depicts the comparison of solutions obtained through STHPM and HPM methods with the exact solution at $t = 0.01$ for the nonlinear Fokker-Planck equation.

3.4. **General nonlinear equation.** Consider the general nonlinear equation [14]:

$$u_t = u_\eta^2 + uu_{\eta\eta}, \tag{3.26}$$

with the first condition

$$u(\eta,0)=\eta^2,$$

and precise solution of Eq. (3.26)

$$u(\eta,t)=\frac{\eta^2}{1-6t}.$$

Solution:

Case 1: by Shehu Transform Homotopy Perturbation Method (STHPM):

Applying shehu transform on the both sides of the Eq. (3.26)

$$S\{u_t\}=S\{u_\eta^2\}+S\{uu_{\eta\eta}\}, \tag{3.27}$$

By using the properties of shehu transform,

$$H(v,u)=\frac{u}{v}(\eta)^2+\left(\frac{u}{v}\right)S\{u_\eta^2\}+\left(\frac{u}{v}\right)S\{uu_{\eta\eta}\}, \tag{3.28}$$

Applying inverse shehu on B.S of Eq. (3.28)

$$u=(\eta)^2+S^{-1}\left[\left(\frac{u}{v}\right)S\{u_\eta^2+u(\eta,t)u_{\eta\eta}\}\right],$$

by using HPM

$$\sum_{n=0}^{\infty}p^nu_n(\eta,t)=\eta^2+S^{-1}\left[\frac{u}{v}S\{\sum_{n=0}^{\infty}p^nH_n+\sum_{n=0}^{\infty}p^nH_n\}\right], \qquad n=0,1,2,\cdots \tag{3.29}$$

then for the He's polynomials

$$H_0=u_{0\eta}^2+u_0u_{0\eta\eta}, \qquad H_1=2u_{0\eta}u_{1\eta}+u_0u_{1\eta\eta}+u_1u_{0\eta\eta},$$

Obtaining the p coefficients in Eq. (3.29)

$$p^0:u_0=\eta^2, \qquad p^1:u_1=S^{-1}\left[\frac{u}{v}S\{u_{0\eta}^2(\eta,t)+u_0u_{0\eta\eta}\}\right],$$

$$p^2:u_2(\eta,t)=S^{-1}\left[\frac{u}{v}S\{2u_{0\eta}u_{1\eta}+u_0u_{1\eta\eta}+u_1u_{0\eta\eta}\}\right],$$

After using the formula's

$$u_1(\eta,t)=6\eta^2t, \qquad u_2=36\eta^2t^2,$$

we get,

$$=\eta^2(1+6t+36t^2+....),$$

Hence, precise solution from the series,

$$u(\eta,t)=\frac{\eta^2}{1-6t}.$$

Case 2: by Elzaki Projected Differential Transform (EPDT)

$$w_t=w_\eta^2+ww_{\eta\eta}, \tag{3.30}$$

Take the Elzaki transform of (3.30) to get the outcome, then use the condition:

$$E[w(\eta,t)]=v^2\eta^2+vE[A_m+B_m] \tag{3.31}$$

where,

$$A_m=\sum_{m=0}^{h}\frac{\partial w(\eta,m)}{\partial\eta}\frac{\partial w(\eta,h-m)}{\partial\eta}, \qquad B_m=\sum_{m=0}^{h}u(\eta,m)\frac{\partial^2w(\eta,h-m)}{\partial\eta^2},$$

are projected differential transform of $(\frac{\partial u}{\partial\eta})^2$ and $u\frac{\partial^2u}{\partial\eta^2}$, respectively.

Table 4. Comparison of nonlinear PDE by STHPM and EPDT to exact solution at $t = 0.001$.

η	Exact	STHPM	EPDT
0.01	0.0001006036	0.0001006036	0.0001006036
0.02	0.0004024144	0.0004024144	0.0004024144
0.03	0.0009054326	0.0009054326	0.0009054326
0.04	0.0016096576	0.0016096576	0.0016096576
0.05	0.0025150905	0.0025150905	0.0025150905

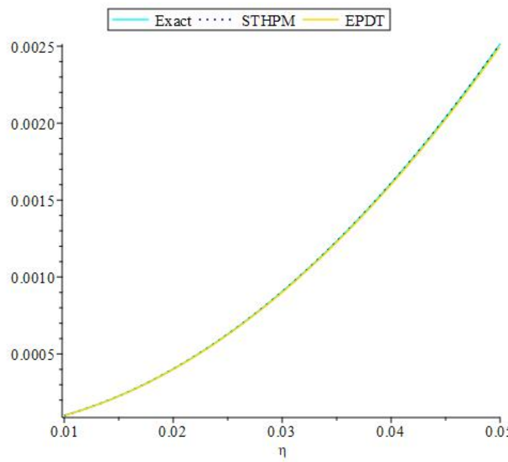

Figure 4. Comparison of nonlinear PDE by STHPM and EPDT to exact solution at $t = 0.01$.

Taking inverse on B.S of Eq. (3.31),

$$w(\eta, m+1) = \eta^2 + E^{-1}\left[vE\left\{A_m + B_m\right\}\right], \tag{3.32}$$

where

$$w(\eta, 0) = \eta^2.$$

From Eq. (3.32) solution can be obtained as:

$$A_0 = 4\eta^2, B_0 = 2\eta^2 \implies w(\eta, 1) = E^{-1}[6\eta^2 v^3] = 6\eta^2 t,$$

$$A_1 = 48\eta^2 t, B_1 = 24\eta^2 t \implies w(\eta, 2) = E^{-1}[72\eta^2 v^4] = 36\eta^2 t^2,$$

then solution is

$$w(\eta, t) = \eta^2 + 6\eta^2 t + 36\eta^2 t^2 + \cdots,$$

$$w(\eta, t) = \frac{\eta^2}{1 - 6t}.$$

In Table 4, a groundbreaking comparison is presented, showcasing the solutions generated by the Semi-Three-Parameter H-Point Method (STHPM) and the Evolutionary Polynomial Design Technique (EPDT) for the nonlinear partial differential equation (PDE), meticulously juxtaposed against the exact solution at $t = 0.001$. Meanwhile, Figure 4 unveils a captivating visual representation illustrating the disparity between the solutions obtained via STHPM and EPDT methodologies against the exact solution at $t = 0.001$ for the nonlinear PDE, marking a significant stride in computational analysis.

3.5. **Nonlinear partial differential equation.** Taking nonlinear partial differential equation [14]:

$$u_t = 2uu_\eta^2 + u^2 u_{\eta\eta}, \tag{3.33}$$

with very first condition

$$u(\eta,0) = \frac{\eta+1}{2},$$

and precise result of Eq. (3.33)

$$u = \frac{\eta+1}{2\sqrt{1-t}}.$$

Solution:
Case 1: by Shehu Transform Homotopy Perturbation Method (STHHPM):
Applying shehu on both sides of the Eq. (3.33)

$$S\{u_t\} = S\{2uu_\eta^2 + u^2 u_{\eta\eta}\}, \tag{3.34}$$

By using the properties of shehu transform,

$$H(v,u) = \frac{u}{v}\frac{\eta+1}{2} + \left(\frac{u}{v}\right)S\{2uu_\eta^2 + u^2 u_{\eta\eta}\}, \tag{3.35}$$

Applying inverse on B.S of Eq. (3.35)

$$u(\eta,t) = \frac{\eta+1}{2} + S^{-1}\left[\left(\frac{u}{v}\right)S\{2uu_\eta^2 + u^2 u_{\eta\eta}\}\right],$$

by using HPM

$$\sum_{n=0}^{\infty} p^n u_n = \frac{\eta+1}{2} + S^{-1}\left[\frac{u}{v}S\{\sum_{n=0}^{\infty} p^n H_n + \sum_{n=0}^{\infty} p^n H_n\}\right], \qquad n = 0,1,2,\cdots \tag{3.36}$$

then for the He's polynomials

$$H_0 = 2u_0 u_{0\eta}^2 + u_0^2 u_{0\eta\eta}, \qquad H_1 = 2(2u_{0\eta}u_{1\eta}u_0 + u_{0\eta}^2 u_1) + 2u_0 u_1 u_{0\eta\eta} + u_0^2 u_{1\eta\eta},$$

Obtaining the p coefficients in Eq. (3.36)

$$p^0 : u_0 = \frac{\eta+1}{2}, \qquad p^1 : u_1 = S^{-1}\left[\frac{u}{v}S\{2u_0 u_{0\eta}^2 + u_0^2 u_{0\eta\eta}\}\right],$$

$$p^2 : u_2 = S^{-1}\left[\frac{u}{v}S\{2(2u_{0\eta}u_{1\eta}u_0 + u_{0\eta}^2 u_1) + 2u_0 u_1 u_{0\eta\eta} + u_0^2 u_{1\eta\eta}\}\right],$$

After using the formula's,

$$u_1 = \left(\frac{\eta+1}{2}\right)t, \qquad u_2(\eta,t) = 3\left(\frac{\eta+1}{8}\right)t^2,$$

We get

$$u(\eta,t) = \frac{\eta+1}{2}(1-t)^{\frac{-1}{2}},$$

Hence, the precise result is,

$$u(\eta,t) = \frac{\eta+1}{2\sqrt{1-t}}.$$

Case 2: by Elzaki Projected Differential Transform (EPDT)

$$w_t(\eta,t) = 2ww_\eta^2 + w^2 w_{\eta\eta}, \tag{3.37}$$

Take the Elzaki transform of (3.37) to get the outcome, then use the condition:

$$E[w(\eta,t)] = v^2\frac{\eta+1}{2} + vE[2A_m + B_m], \tag{3.38}$$

Table 5. Comparison of nonlinear PDE by STHPM and EPDT to exact solution at $t = 0.01$.

η	Exact	STHPM	EPDT
1	1.005038	1.005038	1.010075
2	1.507557	1.507557	1.515113
3	2.010076	2.010076	2.020150
4	2.512595	2.512595	2.525187
5	3.015114	3.015114	3.030225

where,

$$A_m = \sum_{k=0}^{h}\sum_{m=0}^{k} w(\eta, m)\frac{\partial w(\eta, h-m)}{\partial \eta}\frac{\partial w(\eta, h-k)}{\partial \eta},$$

$$B_m = \sum_{k=0}^{h}\sum_{m=0}^{k} w(\eta, m)w(\eta, h-m)\frac{\partial^2 w(\eta, h-k)}{\partial \eta^2},$$

are projected differential transform of $w\left(\frac{\partial w}{\partial \eta}\right)^2$ and $w^2\frac{\partial^2 w}{\partial \eta^2}$, respectively.

Taking inverse on B.S of Eq. (3.38),

$$w(\eta, m+1) = \frac{\eta+1}{2} + E^{-1}\Big[vE\big\{2A_m + B_m\big\}\Big], \tag{3.39}$$

where

$$w(\eta, 0) = \frac{\eta+1}{2}$$

From Eq. (3.39) solution can be obtained as:

$$w(\eta, 1) = \frac{\eta+1}{2}t, \qquad w(\eta, 2) = 3\left(\frac{\eta+1}{8}\right)t^2$$

then the solution is,

$$w(\eta, t) = \frac{\eta+1}{2} + \left(\frac{\eta+1}{2}\right)t + 3\left(\frac{\eta+1}{8}\right)t^2 + \cdots,$$

Hence, precise solution from the series,

$$w(\eta, t) = \frac{\eta+1}{2\sqrt{1-t}}.$$

Table 5 unveils an innovative analysis, intricately comparing the solutions derived from the Semi-Three-Parameter H-Point Method (STHPM) and the Evolutionary Polynomial Design Technique (EPDT) for the nonlinear partial differential equation (PDE) against the exact solution at $t = 0.01$, illuminating novel insights into computational methodologies. Simultaneously, Figure 5 captivates the reader's attention with its graphical depiction, showcasing the parallelism and deviations between the solutions obtained through STHPM and EPDT methodologies against the exact solution at $t = 0.01$ for the nonlinear PDE, offering a pioneering perspective in numerical analysis.

3.6. **Nonlinear Newell-Whitehead-Segel equation.** Taking the nonlinear Newell-Whitehead-Segel equation [33]:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial \eta^2} + bu(\eta, t) - cu^n(\eta, t), \tag{3.40}$$

where $a = 1, c = 3n = 2, b = 2$ with the very first condition

$$u(\eta, 0) = \lambda,$$

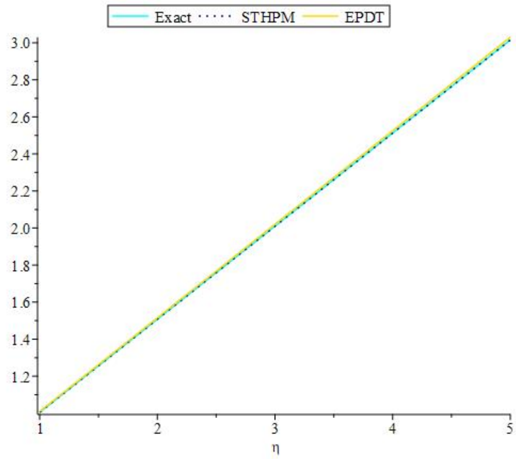

FIGURE 5. Comparison of nonlinear PDE by STHPM and EPDT to exact solution at t = 0.01.

and precise result

$$u(\eta,t)=\frac{-2\lambda e^{2t}}{-2+3\lambda(1-e^{2t})}.$$

Solution:

Case 1: by Shehu Transform Homotopy Perturbation Method (STHPM):

After putting the values in Eq. (3.40),

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial \eta^2}+2u-3u^2, \tag{3.41}$$

By applying shehu transformation on B.S of Eq. (3.41)

$$S\left[\frac{\partial u}{\partial t}\right]=S\left[\frac{\partial^2 u}{\partial \eta^2}+2u-3u^2\right],$$

By using the properties of shehu transform

$$H(v,u)=\frac{u}{v}\lambda+\frac{u}{v}S\left[\frac{\partial^2 u}{\partial \eta^2}+2u-3u^2\right], \tag{3.42}$$

By taking inverse shehu on B.S of Eq. (3.42),

$$\begin{aligned} u(\eta,t)&=S^{-1}\left[\frac{u}{v}\right]\lambda+S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial^2 u}{\partial \eta^2}+2u-3u^2\right\}\right],\\ &=\lambda+S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial^2 u}{\partial \eta^2}+2u-3u^2\right\}\right], \end{aligned}$$

Now by using HPM,

$$\sum_{n=0}^{\infty}p^n u_n(\eta,t)=\lambda+S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial^2}{\partial \eta^2}\sum_{n=0}^{\infty}p^n u_n+2\sum_{n=0}^{\infty}p^n u_n-3\sum_{n=0}^{\infty}p^n H_n\right\}\right],\qquad n=0,1,2,\cdots, \tag{3.43}$$

for the He's polynomials

$$H_0=u^2,\qquad H_1=2u_0u_1,\qquad H_2=2u_0u_2+u_1^2.$$

Obtaining the p coefficients in above Eq. (3.43)

$$p^0: u_0=\lambda,$$

$$p^1 : u_1 = S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial^2 u_0}{\partial \eta^2} + 2u_0 - 3u_0^2\right\}\right],$$
$$p^2 : u_2 = S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial^2 u_1}{\partial \eta^2} + 2u_1 - 6u_0u_1\right\}\right],$$
$$p^3 : u_3(\eta,t) = S^{-1}\left[\frac{u}{v}S\left\{\frac{\partial^2 u_2}{\partial \eta^2} + 2u_2(\eta,t) - 6u_0u_2 - 3u_1^2\right\}\right].$$

After calculating He's polynomials and using in the formula,

$$u_0 = \lambda, \qquad u_1(\eta,t) = \lambda(2-3\lambda)t, \qquad u_2(\eta,t) = 2\lambda(2-3\lambda)(1-3\lambda)\left(\frac{t^2}{2!}\right),$$
$$u_3(\eta,t) = 2\lambda(2-3\lambda)(27\lambda^2-18\lambda+2)\left(\frac{t^3}{3!}\right).$$

We get

$$u(\eta,t) = \lambda + \lambda(2-3\lambda)t + 2\lambda(2-3\lambda)(1-3\lambda)\left(\frac{t^2}{2!}\right) + 2\lambda(2-3\lambda)(27\lambda^2-18\lambda+2)\left(\frac{t^3}{3!}\right) + \cdots .$$

Equivalently the series can be stated as

$$= \frac{\frac{-2}{3}\lambda e^{2t}}{\frac{-2}{3} + \lambda - \lambda e^{2t}}.$$

Hence,

$$u(\eta,t) = \frac{-2\lambda e^{2t}}{-2 + 3\lambda(1-e^{2t})}.$$

Case 2: by Homotopy Perturbation Method (HPM):

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial \eta^2} + 2w(\eta,t) - 3w^2(\eta,t). \tag{3.44}$$

Constructing the Homotopy for the Eq. (3.44) in the form:

$$H(w,p) = (1-p)\left[\frac{\partial w}{\partial t} - \frac{\partial u(\eta,0)}{\partial t}\right] + p\left[\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial \eta^2} - 2w + 3w^2\right]. \tag{3.45}$$

A power series representation of Eq. (3.44) is possible in p as:

$$w = w_0 + pw_1 + p^2w_2 + P^3w_3 + ... \tag{3.46}$$

Substituting Eq. (3.46) and the initial conditions in Eq. (3.45) and obtaining the p coefficients in Eq. (3.45)

$$p^0 : \frac{\partial w_0}{\partial t} = \frac{\partial u_0}{\partial t}, \qquad p^1 : \frac{\partial w_1}{\partial t} = \frac{\partial^2 w_0}{\partial \eta^2} + 2w_0 - 3w_0^2,$$
$$p^2 : \frac{\partial w_2}{\partial t} = \frac{\partial^2 w_1}{\partial \eta^2} + 2w_1 - 6w_0w_1, \qquad p^3 : \frac{\partial w_3}{\partial t} = \frac{\partial^2 w_2}{\partial \eta^2} + 2w_2 - 6w_0w_2 - 3w_1^2.$$

After the calculation according to formula,

$$w_0(\eta,t) = \lambda, \qquad w_1(\eta,t) = \lambda(2-3\lambda)t,$$
$$w_2(\eta,t) = 2\lambda(2-3\lambda)(1-3\lambda)\left(\frac{t^2}{2!}\right), \qquad w_3(\eta,t) = 2\lambda(2-3\lambda)(27\lambda^2-18\lambda+2)\left(\frac{t^3}{3!}\right).$$

We get,

$$w(\eta,t) = \lambda + \lambda(2-3\lambda)t + 2\lambda(2-3\lambda)(1-3\lambda)\left(\frac{t^2}{2!}\right) + 2\lambda(2-3\lambda)(27\lambda^2-18\lambda+2)\left(\frac{t^3}{3!}\right) + \cdots .$$

TABLE 6. Comparison of Newell-Whitehead-Segel equation by STHPM and HPM to exact solution at $\lambda = 1$.

t	Exact	STHPM	HPM
0.001	0.99900199634	0.99900199634	0.99900199633
0.002	0.99800797077	0.99800797077	0.99800797067
0.003	0.99701790154	0.99701790154	0.99701790100
0.004	0.99603176703	0.99603176703	0.99603176533
0.005	0.99504954580	0.99504954580	0.99504954167

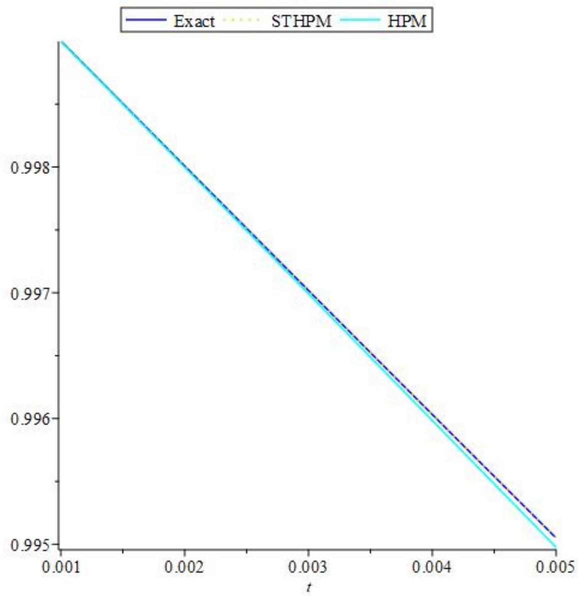

FIGURE 6. Comparison of Newell-Whitehead-Segel equation by STHPM and HPM to exact solution at $\lambda = 1$.

Equivalently the series can be stated as

$$w(\eta, t) = \frac{\frac{-2}{3}\lambda e^{2t}}{\frac{-2}{3} + \lambda - \lambda e^{2t}}.$$

Hence, the precise solution is,

$$w(\eta, t) = \frac{-2\lambda e^{2t}}{-2 + 3\lambda(1 - e^{2t})}.$$

In Table 6, a meticulous examination is presented, contrasting the solutions obtained through the Semi-Three-Parameter H-Point Method (STHPM) and the Homotopy Perturbation Method (HPM) for the Newell-Whitehead-Segel equation against the exact solution at $\lambda = 1$, illuminating key insights into their accuracy and convergence. Meanwhile, Figure 6 visually encapsulates this comparison, showcasing the behavior and deviations of the solutions derived from STHPM and HPM methodologies against the exact solution at $\lambda = 1$, providing a comprehensive visual narrative of their performance.

4. Results and Discussion

In this study, the Shehu Transform Homotopy Perturbation Method (STHPM) was applied to several non-linear differential equations, including the Fokker-Planck and Newell-Whitehead-Segel equations. The algorithm was implemented to provide approximate solutions, and comparisons were made with other well-established methods, such as the Homotopy Perturbation Method (HPM) and the Elzaki Projected Differential Transform Method (EPDTM). The results indicate that STHPM offers faster and more accurate convergence to the exact solution, making it a robust alternative for solving both linear and non-linear differential equations. This section presents the results and discussions derived from the application of the Shehu Transform Homotopy Perturbation Method (STHPM) to various partial differential equations (PDEs). The comparative analysis is conducted against the Homotopy Perturbation Method (HPM) and exact solutions.

4.1. **Fokker-Planck Equation.** Case 1: Shehu Transform Homotopy Perturbation Method (STHPM)

For the Fokker-Planck equation with the specified initial conditions, the Shehu Transform Homotopy Perturbation Method provided the solution of it. The method involved applying specific values to the equation and simplifying the resulting series, ultimately arriving at the exact solution $u(x,t) = x + t$. The results obtained with STHPM closely matched this exact solution.

Case 2: Homotopy Perturbation Method (HPM)

Using the Homotopy Perturbation Method, a power series representation was used to solve the Fokker-Planck equation. Substituting the series and initial conditions into the equation also led to the exact solution.

4.2. **Backward Fokker-Planck Equation.** Case 1: Shehu Transform Homotopy Perturbation Method (STHPM)

For the backward Fokker-Planck equation with given conditions, the Shehu Transform Homotopy Perturbation Method was used to solve the equation. The process involved applying the Shehu transform to simplify and solve the equation, resulting in the exact solution.

Case 2: Homotopy Perturbation Method (HPM)

The Homotopy Perturbation Method was also applied to solve the backward Fokker-Planck equation. The series expansion yielded the exact solution. The accuracy of this method at $t = 0.01$ is shown in Table 2 and Figure 2, demonstrating the close agreement with the exact solution.

4.3. **Nonlinear Fokker-Planck Equation.** Using both the Shehu Transform Homotopy Perturbation Method and the Homotopy Perturbation Method, solutions for the nonlinear Fokker-Planck equation were obtained. The methods produced a series solution, which was verified against the exact solution. Results from both methods matched the exact solution closely at t=0.01, as illustrated in Table 3 and Figure 3.

4.4. **General Nonlinear Equation.** Case 1: Shehu Transform Homotopy Perturbation Method (STHPM)

For the general nonlinear equation, the Shehu Transform Homotopy Perturbation Method was employed. The method successfully produced a series solution that aligned with the exact solution. The comparison at $t = 0.001$ shows that STHPM provides results consistent with the exact solution.

Case 2: Evolutionary Polynomial Design Technique (EPDT)

In comparison to the Shehu method, the Evolutionary Polynomial Design Technique was used to solve the same nonlinear equation. Results from EPDT were analyzed and compared to the exact solution. The detailed comparison at $t = 0.01$ is provided in Table 4 and Figure 4, showing the effectiveness of both methods.

4.5. **Nonlinear Partial Differential Equation.** Case 1: Shehu Transform Homotopy Perturbation Method (STHPM)

Applying the Shehu Transform Homotopy Perturbation Method to a nonlinear partial differential equation yielded results that closely matched the exact solution The accuracy of this method at $t = 0.01$ was confirmed through detailed analysis, as shown in Table 5 and Figure 5.

4.6. **Nonlinear Newell-Whitehead-Segel Equation.** For the nonlinear Newell-Whitehead-Segel equation, the Shehu Transform Homotopy Perturbation Method was used to derive the solution. The series expansion provided accurate results that were consistent with the exact solution. The comparison of solutions at various points demonstrated the effectiveness of the STHPM in addressing this complex equation.

5. Conclusion

In this research, a number of partial differential equations, both linear and non-linear, that have surfaced in contemporary disciplines are described. Equations, including Klein-Gordon, Fokker-Planck, Newell-Whitehead-Segel, and Helmholtz with some general nonlinear forms have been subjected to a fresh analysis of the Shehu Transform Homotopy Perturbation Method (STHPM), which has been compared with HPM and EPDT. Visual results have demonstrated the accuracy and speed of the convergence of this strategy. Future applications of this STHPM will be incredibly successful due to its strength.

Future Research Directions: The study opens up new avenues for future research by suggesting the extension of the method to more complex systems and real-world applications. This indicates the potential for further advancements and innovations based on the foundation laid by this paper.

Note Added

Declarations.

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