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An accurate finite-difference scheme for the numerical solution of a fractional differential equation

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Abstract

In this article, a steady-state fractional-order boundary-value problem is considered with a fractional convection term. The highest-order derivative term involves a mixed-fractional derivative, which appears as a combination of a first-order classical derivative and a Caputo fractional derivative. We propose an $L1$ scheme over a uniform mesh for the numerical solution of the fractional differential equation. With the help of a properly chosen barrier function, we discuss error analysis and prove that the proposed method converges with almost first-order. The proposed scheme is also applied to a semilinear fractional differential equation. Numerical experiments are presented to validate the proposed method.

Keywords. Mixed-fractional derivative, Fractional differential equation, Fractional-convection term, $L1$ method, Discrete comparison principle, Stability, Convergence.

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1. Introduction

Fractional differential equations (FDEs) are the generalized version of classical differential equations that include arbitrary (or non-integer) order derivatives. It has attracted many researchers for the past few years because of its applications in various fields of science and engineering, such as plasma physics [7], viscoelasticity [3], the Lévy process [10], finance [19], etc.

In this manuscript, we consider a steady-state fractional-convection-diffusion-type two-point boundary-value problem with a reaction term:

$$\begin{cases} -D(D_C^{\alpha-\beta}u(x)) + a(x)D_C^{\beta}u(x) + r(x)u(x) = f(x), \quad x \in \Omega = (0,1), \\ D_C^{\alpha-\beta}u(0) = 0, \quad u(1) + \sigma D_C^{\alpha-\beta}u(1) = \gamma, \end{cases} \tag{1.1}$$

where $0 < \alpha - \beta < \beta \leq 1 < \alpha < 2$ and $D = d/dx$ represents the first-order classical derivative operator with respect to x. Consider functions $a, r,$ and f, which belongs to $\mathcal{C}(\overline{\Omega})$. Additionaly, let σ (where $\sigma(\geq 0)$) and γ be constants.

The mixed-fractional differential operator $D(D_C^{\alpha-\beta})$, considered in this paper, is the combination of the first-order classical derivative operator and the $(\alpha-\beta)^{th}-$ order Caputo derivative operator. This type of mixed-fractional derivative was first introduced by Patie and Simmon [17] in the study of asymmetric α-stable Lévy processes, and they used the derivative operator as the infinitesimal generator of the spectrally positive (or spectrally negative) α-stable Lévy process reflected at its running supremum. In [27], this mixed-fractional derivative is termed a conservative Caputo derivative. Gracia et al. [8] proposed a finite difference scheme for the steady-state diffusion-convection-reaction-type fractional differential equation containing this type of mixed-fractional derivative with $\beta = 1$, and they named it the Riemann-Liouville-Caputo (RLC) derivative. Also, Gracia and Stynes have solved the time-dependent

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parabolic problem in [9], containing the RLC-type spatial fractional derivative, numerically by a finite difference scheme.

The fractional-convection term was first studied by Li et al. [14] in the context of the failure of the convection process to obey the power-law distribution in the meantime. This concept of the fractional-convection term has found applications in various fields such as porous media flow, where non-local transport phenomena are prevalent, and in biological systems, where anomalous diffusion behavior is observed. The fractional-convection term allows for a more accurate description of these complex transport processes.

For all $x \in (0,1]$, the Caputo fractional derivative with order $\delta \in (0,1)$ of sufficiently smooth functions is defined as follows [2, 13]:

$$D_C^{\delta} u(x) = \frac{1}{\Gamma(1-\delta)} \int_0^x (x-t)^{-\delta} u'(t)\, dt,$$

where $u'(t)$ is the first-order classical derivative of $u(t)$.

Following [8, 9, 17], the mixed-fractional derivative whose order lies in $(1,2)$, defined by

$$D(D_C^{\alpha-\beta})\, u(x) = \frac{d}{dx} \frac{1}{\Gamma(1-\alpha+\beta)} \int_0^x (x-t)^{\beta-\alpha} u'(t)\, dt,$$

provided $u' \in A^2(\overline{\Omega})$ where the space $A^2(\overline{\Omega})$ is defined by

$$A^2(\overline{\Omega}) = \left\{ g \in \mathcal{C}^1(\overline{\Omega}) : g' \text{ is absolutely continuous on } \overline{\Omega} \right\}.$$

Many a time, the existence of a solution to a differential equation does not imply one can find the exact solution. To overcome this difficulty, researchers have developed various types of numerical methods. Stynes and Gracia [22] have used the L_1 method, based on the finite difference scheme, to solve a Caputo-type FDE. Zhang [29] has applied the finite difference method to solve fractional-order partial differential equation (PDE). Seal and Natesan [20] have studied the error estimate of the second-order spline method to solve a fractional boundary value problem with nonlocal boundary conditions.

Ervin and Roop [5] applied the finite element method (FEM) to solve the fractional advection-dispersion equation. Also, Ford et al. [6] used a fully discrete scheme by using FEM over a time fractional PDE, which includes a Riemann-Liouville-type time fractional derivative. Then a modified version of FEM called the weak Galerkin finite element method (WG-FEM) has attracted some researchers. In [28], Wang et al. established a scheme based on WG-FEM for solving a class of generalized time-dependent fractional Burger's equation. Seal et al. [21] have applied a dimensional-splitting WG-FEM to solve a 2D time-fractional diffusion equation and discussed stability and error analysis of their proposed scheme.

Some researchers were interested in applying the meshless method for solving FDEs. Mardani et al. [16] solved a time fractional advection-diffusion equation with a meshless method based on the moving least squares method. Also, a meshless method in reproducing kernel space was used by Du et al. [4] to solve an advection-diffusion equation that includes a variable-order Caputo-type time fractional derivative.

To the best of our knowledge, this is the first article that studies the model problem defined by Equation (1.1), incorporating a fractional convection term, and solved by using the well-known $L1$ method. This type of boundary value problem, presented in (1.1), represents a steady-state scenario, where the goal is to model systems that have reached equilibrium or exhibit time-independent behavior. In engineering, such steady-state models are crucial for understanding the long-term behavior of materials and structures under constant loads, especially when those materials exhibit fractional viscoelasticity or anomalous diffusion. Also, this formulation is essential for steady-state wave propagation problems in media with non-local effects or memory, capturing the system's equilibrium state after transient dynamics have subsided. In control systems, the steady-state formulation is employed to design controllers that ensure the system reaches and maintains a desired equilibrium state, even in the presence of complex fractional dynamics. Hence, the steady-state fractional boundary value problem is pivotal for understanding long-term and equilibrium behavior in various applied domains. Further, we study the stability of the proposed numerical scheme and derive the error estimate by using a discrete barrier function. In addition, we exert the proposed scheme for a semilinear FDE. To validate the theoretical error estimate and order of convergence, some numerical experiments are carried out.

This paper is organized in the following manner: Section 2 discusses some useful results and properties as well as the existence and uniqueness theorem of the given model problem. Section 3 starts with the discretization of the model problem, which is followed by the discrete comparison principle. In Section 4, the truncation error bound and the convergence of the computed solution to the exact solution with the help of a properly chosen discrete barrier function are established. In Section 5, the proposed method is applied to solve a semilinear FDE. Section 6 presents numerical results, followed by conclusions in Section 7.

Notation: C represents a generic constant that takes different positive values according to its presence in the required place. $\|\cdot\|_\infty$ is the maximum norm on $\overline{\Omega}$. $\mathcal{C}^k(\overline{\Omega})$ is the space of k times continuously differentiable functions defined on $\overline{\Omega}$.

2. Preliminary Results and Properties

This section starts with an important lemma that shows the equivalency between the first-order classical derivative and the $(\alpha-\beta)^{th}-$ order Caputo derivative at the boundary condition for $x=0$.

Lemma 2.1. *Suppose the functions $a, r, f \in \mathcal{C}(\overline{\Omega})$ along with the presence of the existence of $D(D_C^{\alpha-\beta}u(x))$ where $u(x)$ is the solution of the model problem (1.1). Then the left-side boundary condition is equivalent to $u'(0)=0$.*

Proof. This can be proved in the similar way as given in [8, Lemma 2.1]. □

Next, we describe the reason behind the choice of the left boundary condition as $D_C^{\alpha-\beta}u(0)=0$ for our problem by using the Sumudu transform.

2.1. **Necessity of the left boundary condition.** We check the requirement of the left boundary condition given in the FDE (1.1) by using the Sumudu transform [11] on both sides of (1.1).

Let us consider a special case of (1.1), such as

$$-D(D_C^{\alpha-\beta}u(x)) + aD_C^{\beta}u(x) + ru(x) = f(x), \quad \text{on } \Omega, \tag{2.1}$$

where the coefficients a and r are considered as constants. The Sumudu transform of $D_C^{\alpha-\beta}u(x)$ is given by

$$\mathcal{S}\left[D_C^{\alpha-\beta}u(x);p\right] = p^{-(\alpha-\beta)}\mathcal{S}[u;p] - p^{-(\alpha-\beta)}u(0).$$

And the Sumudu transform of the first-order derivative of the mixed-fractional derivative $D\left(D_C^{\alpha-\beta}u(x)\right)$ is

$$\mathcal{S}\left[D\left(D_C^{\alpha-\beta}u(x)\right);p\right] = \frac{\mathcal{S}[u;p]}{p^{\alpha-\beta+1}} - \frac{u(0)}{p^{\alpha-\beta+1}} - \frac{D_C^{\alpha-\beta}u(0)}{p}.$$

Thus, applying the Sumudu transform on both sides of (2.1), one gets

$$-\frac{\mathcal{S}[u;p]}{p^{\alpha-\beta+1}} + \frac{u(0)}{p^{\alpha-\beta+1}} + \frac{D_C^{\alpha-\beta}u(0)}{p} + a\left[\frac{\mathcal{S}[u;p]}{p^{\beta}} - \frac{y(0)}{p^{\beta}}\right] + r\,\mathcal{S}[u;p] = \mathcal{S}[f;p],$$

which implies

$$\mathcal{S}[u;p] = u(0) + \frac{D_C^{\alpha-\beta}u(0)}{p(p^{-(\alpha-\beta+1)} - ap^{-\beta} - r)} \quad + \quad \frac{r\,u(0)}{p^{-(\alpha-\beta+1)} - ap^{-\beta} - r} - \frac{\mathcal{S}[f;p]}{p^{-(\alpha-\beta+1)} - ap^{-\beta} - r}. \tag{2.2}$$

Now, we state the following result [1]:

$$\mathcal{S}^{-1}\left\{\frac{u^{-\rho}}{u^{-\alpha} + au^{-\beta} + b}\right\} = \sum_{n=0}^{\infty}(-a)^n x^{(\alpha-\beta)n+\alpha-\rho}\, E^{n+1}_{\alpha,\alpha+(\alpha-\beta)n-\rho+1}(-bx^{\alpha}),$$

where $E^{\eta}_{\zeta,\delta}$ is the three-parameter Mittag-Leffler function defined by

$$E^{\eta}_{\zeta,\delta}(z) = \frac{1}{\Gamma(\eta)} \sum_{k=0}^{\infty} \frac{\Gamma(\eta+k) z^k}{k!\Gamma(\zeta k+\delta)}, \quad \text{for} \quad \eta, \zeta, \delta, z \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}(\zeta) > 0.$$

Therefore, applying the inverse Sumudu transform on both sides of (2.2) and using the above result, one can obtain

$$u(x) = \Psi(x) D_C^{\alpha-\beta} u(0) + u(0)(1 + r\,\Phi(x)) - f * \Phi(x), \tag{2.3}$$

where

$$\Psi(x) = \sum_{n=0}^{\infty} a^n x^{(\alpha-2\beta+1)n+\alpha-\beta}\, E^{n+1}_{\alpha-\beta+1,\alpha-\beta+1+(\alpha-2\beta+1)n}(r x^{\alpha-\beta+1}),$$

and

$$\Phi(x) = \sum_{n=0}^{\infty} a^n x^{(\alpha-2\beta+1)n+\alpha-\beta+1}\, E^{n+1}_{\alpha-\beta+1,\alpha-\beta+2+(\alpha-2\beta+1)n}(r x^{\alpha-\beta+1}).$$

From (2.3), one can judge that the solution is only continuous and a troublesome singularity exists in the solution u at $x = 0$ when $D_C^{\alpha-\beta} u(0) \neq 0$. Thus, for the solution to be in $\mathcal{C}^1(\overline{\Omega})$, the left boundary condition should be $D_C^{\alpha-\beta} u(0) = 0$.

Remark 2.2. If we consider the reaction coefficient $r = 0$ and the function $f(x)$ is taken to be constant in the equation (2.1), then the solution will be

$$\begin{aligned} u(x) &= u(0) + D_C^{\alpha-\beta} u(0) x^{\alpha-\beta} E_{1+\alpha-2\beta,\,\alpha-\beta+1}\left(a x^{\alpha+1-2\beta}\right) \\ &\quad - f x^{\alpha-\beta+1} E_{1+\alpha-2\beta,\,\alpha-\beta+2}\left(a x^{\alpha+1-2\beta}\right), \end{aligned} \tag{2.4}$$

where $E_{\zeta,\delta}$ is the two-parameter Mittag-Leffler function defined by

$$E_{\zeta,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\zeta k+\delta)}, \quad \text{for} \quad \zeta, \delta, z \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}(\zeta) > 0.$$

Remark 2.3. The solutions given by (2.3) and (2.4) include $y(0)$, which can be calculated using the right-hand boundary condition due to the continuity of the solution in the domain $\overline{\Omega}$.

Remark 2.4. For $\beta = 1$, the choice of the left boundary condition for the existence of a continuously differentiable solution has been shown in [8, Subsection 2.1.] by considering a special case with constant coefficient and constant function f along with use of the Laplace transform in the corresponding model problem.

Let us consider the space $\mathcal{C}^{k,\eta}(0,1]$, for each positive integer k and $-\infty < \eta < 1$, which contains functions $y \in \mathcal{C}(\overline{\Omega})$ such that the functions $y \in \mathcal{C}^k(0,1]$ and satisfy the bounds

$$|y^{(i)}(x)| \leq \begin{cases} C, & \text{if} \quad i < 1-\eta, \\ C(1 + |\ln(x)|), & \text{if} \quad i = 1-\eta, \\ C x^{1-\eta-i}, & \text{if} \quad i > 1-\eta, \end{cases} \tag{2.5}$$

for $x \in (0,1]$ and $i = 1, 2, \ldots, k$.

For our problem (1.1), the result from [18, Theorem 2.1] gives the following assumption and theorem for the existence and uniqueness of the solution.

Assumption 2.5. *For $f \equiv 0$ and $\gamma \equiv 0$, the FDE (1.1) has the trivial solution $u \equiv 0$.*

Theorem 2.6. *Let $a, r, f \in \mathcal{C}^{k,\eta}(0,1]$ for some integer $k \geq 2$ and some $\eta \in (-\infty, 1)$. Then the FDE (1.1) has a unique solution $u \in \mathcal{C}^1(\overline{\Omega})$ with $D(D_C^{\alpha-\beta} u) \in \mathcal{C}^{k,\lambda}(0,1]$ where $\lambda := \max\{\eta, 2\beta - \alpha\}$.*

The bound of the derivatives of the solution $u(x)$ of the model problem (1.1) by summoning the idea from [22, Corollary 3.1] and [8, Corollary 2.1] can be given as

Corollary 2.7. *Let $a, r, f \in \mathcal{C}^{k,\eta}(0,1]$ for some positive integer k and some $\eta \in (-\infty, 1)$ with $\eta \leq 2\beta - \alpha$. Then from Theorem 2.6, the FDE (1.1) has a unique solution u where $u \in \mathcal{C}^{k+1}(0,1]$ and there exists a constant C such that the solution u satisfies*

$$\left|u^{(i)}(x)\right| \leq \begin{cases} C, & \text{for} \quad i = 0, \\ Cx^{\alpha-\beta+1-i}, & \text{for} \quad i = 1, 2, \ldots, k+1, \quad \text{and} \quad x \in (0,1]. \end{cases} \tag{2.6}$$

Proof. It can be proved in the same way as given in [8, Corollary 2.1]. □

Remark 2.8. Gracia et al. proved a similar kind of bound in [8, Corollary 2.1], but with $\beta = 1$ and a different boundary condition at $x = 1$.

3. Discretization and Comparison Principle

This section is addressed to study the numerical technique to discretize the given model problem (1.1) and to establish the discrete comparison principle for the proposed method.

Let N be a positive integer. We employ a uniform mesh characterized by a mesh length $h = 1/N$ and mesh points $x_j = jh = j/N$ to discretize the domain $\overline{\Omega}$, where $j = 0, 1, \ldots, N$. Then the discretized form of (1.1) is

$$\begin{cases} \text{Find } \{U_j\}_{j=0}^{N} \text{ such that} \\ L_N U_j := -D^+(D_{C,L1}^{\alpha-\beta} U_j) + a_j D_{C,L1}^{\beta} U_j + r_j U_j = f_j, \quad \text{for } j = 1, 2, \ldots, N-1, \\ -D^+ U_0 = 0, \quad U_N + \sigma D_{C,L1}^{\alpha-\beta} U_N = \gamma, \end{cases} \tag{3.1}$$

where $a_j := a(x_j)$ and similar expression for r_j and f_j also, while $D^+U_j = (U_{j+1} - U_j)/h$ denotes the standard forward difference formula.

Here $D_{C,L1}^{\alpha-\beta} U_j$ and $D_{C,L1}^{\beta} U_j$, the $L1-$ discretization of $D_C^{\alpha-\beta} u(x_j)$ and $D_C^{\beta} u(x_j)$ respectively, are given by

$$\begin{aligned} D_{C,L1}^{\alpha-\beta} U_j &:= \frac{1}{\Gamma(1-\alpha+\beta)} \sum_{k=0}^{j-1} \int_{x_k}^{x_{k+1}} (x_j - t)^{\beta-\alpha} \frac{U_{k+1} - U_k}{h}\, dt \\ &= \frac{h^{\beta-\alpha}}{\Gamma(2-\alpha+\beta)} \sum_{k=0}^{j-1} (U_{k+1} - U_k)\, d_{j-k}, \quad j = 1, 2, \ldots, N, \end{aligned} \tag{3.2}$$

where

$$d_k = \begin{cases} k^{1-\alpha+\beta} - (k-1)^{1-\alpha+\beta}, & \text{for} \quad k = 1, 2, \ldots, N, \\ 0, \quad \text{for} \quad k \leq 0, \end{cases} \tag{3.3}$$

and

$$D_{C,L1}^{\beta} U_j = \frac{h^{-\beta}}{\Gamma(2-\beta)} \sum_{k=0}^{j-1} (U_{k+1} - U_k)\, b_{j-k}, \tag{3.4}$$

where

$$b_k = \begin{cases} k^{1-\beta} - (k-1)^{1-\beta}, & \text{for} \quad k = 1, 2, \ldots, N, \\ 0, \quad \text{for} \quad k \leq 0. \end{cases} \tag{3.5}$$

Therefore, we have

$$\begin{aligned}
-D^{+}(D^{\alpha-\beta}_{C,L1}U_j) &= -\frac{D^{\alpha-\beta}_{C,L1}U_{j+1} - D^{\alpha-\beta}_{C,L1}U_j}{h} \\
&= -\frac{h^{-(1+\alpha-\beta)}}{\Gamma(2+\beta-\alpha)}\left[\sum_{k=0}^{j}(U_{k+1}-U_k)\,d_{j-k+1} - \sum_{k=0}^{j-1}(U_{k+1}-U_k)\,d_{j-k}\right] \\
&= -\frac{h^{-(1+\alpha-\beta)}}{\Gamma(2+\beta-\alpha)}\left[U_0(d_j-d_{j+1}) + \sum_{k=1}^{j}(d_{j-k+2}-2d_{j-k+1}+d_{j-k})U_k + U_{j+1}d_1\right],
\end{aligned}$$

for $j = 1, 2, \dots\ N-1$.

Lemma 3.1. *The following relations hold for the coefficients d_j and b_j [8, Section 3],*

$$(i)\ d_{k+1} < d_k \quad \textit{and} \quad b_{k+1} < b_k, \quad \textit{for all integers} \quad k \geq 1, \tag{3.6}$$

$$(ii)\ d_{j-k+2} - 2d_{j-k+1} + d_{j-k} > 0, \quad \textit{for} \quad 1 \leq j \leq N-1, \textit{and} \quad k \in \{1, \dots, j-1, j+1\}. \tag{3.7}$$

We next discuss the discrete comparison principle that plays an important role showing the convergence of the computed solution to the exact solution.

3.1. **Discrete comparison principle.**

Lemma 3.2. *Assume that the coefficients in our problem (1.1) satisfy (i) $a,\ r,\ f \in \mathcal{C}^{k,\eta}(0,1] \subset \mathcal{C}(\overline{\Omega})$ and (ii) $r(x) \geq 0$ for $x \in \overline{\Omega}$.*

Let us consider a mesh function $\{Z_j\}_{j=0}^{N}$ that satisfies

$$-D^{+}Z_0 \geq 0, \quad Z_N + \sigma D^{\alpha-\beta}_{C,L1}Z_N \geq 0, \quad \textit{and} \quad L_N Z_j \geq 0, \quad \textit{for } j = 1, 2, \dots, N-1.$$

Let the mesh width h satisfy

$$h^{1+\alpha-2\beta} < E(\alpha,\beta)\min\left\{\left(\frac{d_j - d_{j+1}}{b_j}\right)_{j=1}^{N-1}, \left(\frac{d_{j+2} - 2d_{j+1} + d_j}{b_j - b_{j+1}}\right)_{j=1}^{N-2}\right\}, \tag{3.8}$$

where $E(\alpha,\beta) = \dfrac{\Gamma(2-\beta)}{\|a\|_\infty \Gamma(2-\alpha+\beta)}$. Then $Z_j \geq 0$, for $j = 0, 1, \dots, N$.

Proof. It can be observed from (3.1) that after some steps one can obtain

$$\begin{aligned}
&Z_0\left[\frac{d_{j+1}-d_j}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} - \frac{a_j b_j}{h^{\beta}\Gamma(2-\beta)}\right] + \sum_{k=1}^{j+1}\left[\frac{-(d_{j-k+2}-2d_{j-k+1}+d_{j-k})}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)}\right. \\
&\quad \left. + \frac{a_j(b_{j-k+1}-b_{j-k})}{h^{\beta}\Gamma(2-\beta)}\right] Z_k + r_j Z_j = f_j, \quad \text{where} \quad j = 1, 2, \dots N-1.
\end{aligned}$$

Let $P = (p_{j,k})_{j,k=0}^{N}$ be the $(N+1)\times(N+1)$ matrix associated to the discretization (3.1). Then we have a linear system $P\overrightarrow{Z} = \overrightarrow{f}$ where $\overrightarrow{Z} = (Z_0, Z_1, \dots, Z_N)$ and $\overrightarrow{f} = (f_0, f_1, \dots, f_N)$, with $f_0 = 0$ and $f_N = \gamma$. By the hypothesis

$$P\overrightarrow{Z} \geq 0. \tag{3.9}$$

Entries of the 0^{th} row of the matrix P are

$$p_{0,0} = \frac{1}{h}, \quad p_{0,1} = -\frac{1}{h}, \quad p_{0,k} = 0, \quad 1 < k \leq N.$$

For $j = 1, 2, \dots N-1$, the entries of j^{th} row are

$$
\begin{aligned}
p_{j,0} =& \frac{d_{j+1} - d_j}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} - \frac{a_j b_j}{h^{\beta}\Gamma(2-\beta)}, && (3.10)\\
p_{j,k} =& \frac{-(d_{j-k+2} - 2d_{j-k+1} + d_{j-k})}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} + \frac{a_j(b_{j-k+1} - b_{j-k})}{h^{\beta}\Gamma(2-\beta)}, \qquad \text{for} \quad k = 1, 2, \dots, j-1, j+1, && (3.11)\\
p_{j,j} =& \frac{2d_1 - d_2}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} + a_j \frac{b_1}{h^{\beta}\Gamma(2-\beta)} + r_j, && (3.12)\\
p_{j,k} =& 0, \quad \text{for} \quad k = j+2, j+3, \dots, N. && (3.13)
\end{aligned}
$$

The N^{th} row of the matrix P is

$$
\begin{aligned}
&p_{N,0} = -\sigma M d_N, \quad p_{N,k} = \sigma M(d_{N-k+1} - d_{N-k}), \quad \text{for} \quad k = 1, 2 \dots, N-1,\\
&p_{N,N} = 1 + \sigma M d_1, && (3.14)
\end{aligned}
$$

where $M = h^{\beta-\alpha}/\Gamma(2-\alpha+\beta)$. The entries given in (3.14) satisfy $p_{N,0} < 0$, $p_{N,k} < 0$, and $p_{N,N} > 0$.

We now prove $P^{-1} > 0$ by considering two cases for the different signs of the coefficient function $a(x)$.

First, considering the case $a \geq 0$ and recalling Lemma 3.1, one can easily obtain that the off-diagonal entries of the matrix P are non-positive and the diagonal entries are strictly positive i.e., $p_{j,k} \leq 0$ for $k = 0, 1, \dots, j-1, j+1, \dots, N$ and $p_{j,j} > 0$ where $0 \leq j \leq N$.

Now, include the case $a < 0$. Denote $\mu = \Gamma(2-\beta)/\Gamma(2-\alpha+\beta)$. For $j = 1, 2, \dots, N-1$, one can rewrite (3.10) as

$$
\begin{aligned}
p_{j,0} &= -\frac{d_j - d_{j+1}}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} + \frac{(-a_j)b_j}{h^{\beta}\Gamma(2-\beta)}\\
&= \frac{(-a_j)b_j}{h^{\alpha-\beta+1}\Gamma(2-\beta)}\left[h^{\alpha-2\beta+1} - \mu\frac{d_j - d_{j+1}}{(-a_j)b_j}\right] < 0,
\end{aligned}
$$

because of the condition (3.8) on mesh width h.

From (3.11), one can write

$$
\begin{aligned}
p_{j,k} &= \frac{-(d_{j-k+2} - 2d_{j-k+1} + d_{j-k})}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} - \frac{-(a_j)(b_{j-k+1} - b_{j-k})}{h^{\beta}\Gamma(2-\beta)}\\
&= \frac{(-a_j)(b_{j-k} - b_{j-k+1})}{h^{\alpha-\beta+1}\Gamma(2-\beta)}\left[h^{\alpha-2\beta+1} - \mu\frac{d_{j-k+2} - 2d_{j-k+1} + d_{j-k}}{(-a_j)(b_{j-k} - b_{j-k+1})}\right]. && (3.15)
\end{aligned}
$$

For $k = 1, 2 \dots, j-1$, one has $1 \leq j-k \leq j-1$ with $j = 2, 3, \dots, N-1$. Thus, using Eq. (3.8) along with Lemma 3.1, it is clear to notice in (3.15) that $p_{j,k} < 0$, and for $k = j+1$, we have $p_{j,j+1} = -d_1/(h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)) < 0$, where $1 \leq j \leq N-1$.

It remains to discuss the bound for the diagonal entries. The assumption $r_j > 0$ is already mentioned in the statement of the lemma. The sum of the first two terms in (3.12) can be rewritten as

$$
\frac{2d_1 - d_2}{h^{\alpha-\beta+1}\Gamma(2-\alpha+\beta)} - (-a_j)\frac{b_1}{h^{\beta}\Gamma(2-\beta)}, \quad \text{for} \quad j = 1, 2, \dots, N-1,
$$

which shows that $p_{j,j} > 0$ owing to the mesh condition (3.8) and the relation $2d_1 - d_2 > d_1 - d_2$.

Thus, the matrix P has positive diagonal entries and non-positive off-diagonal entries.

Now,

$$
\sum_{k=1}^{N} |p_{0,k}| = \frac{1}{h} = |p_{0,0}|, \quad \sum_{k=0}^{N-1} |p_{N,k}| = \sigma M d_1 < |p_{N,N}|, \quad \text{and} \quad \sum_{\substack{k=0\\k\neq j}}^{N} |p_{j,k}| < |p_{j,j}|.
$$

Consequently, P is an irreducibly diagonally dominant matrix, and we have from [26, Corollary 3.20], $P^{-1} > 0$. Thus, from (3.9), we get $\overrightarrow{Z} \geq 0$, and hence the result follows. □

In the above lemma, we have discussed the discrete comparison principle for L_N. Proceeding similarly to [8, Lemma 3.2], one can prove the discrete comparison principle for $D^{\alpha-\beta}_{C,L1}$.

4. Error Analysis

In this section, our aim is to estimate the error by identifying truncation errors and their associated bounds, along with studying the convergence of the computed solution towards the exact solution. Here $u(x_j)$ is the exact solution and U_j is the computed solution of the FDE (1.1) at the point $x_j, j = 0, 1, \dots, N$.

4.1. **Truncation error.** The truncation errors of the discretization (3.1) of the FDE (1.1) are given by

$$\begin{aligned} L_N(u(x_j) - U_j) &= -D^+ D^{\alpha-\beta}_{C,L1}(u(x_j) - U_j) + a_j D^{\beta}_{C,L1}(u(x_j) - U_j) + r_j(u(x_j) - U_j) \\ &= -(D^+ D^{\alpha-\beta}_{C,L1} - DD^{\alpha-\beta}_{C})u(x_j) + a_j(D^{\beta}_{C,L1} - D^{\beta}_{C})u(x_j), \quad \text{for} \quad 0 < x_j < 1, \end{aligned} \tag{4.1}$$

and for the boundary conditions

$$-D^+(u(x_0) - U_0) = \frac{u(0) - u(h)}{h},$$

and

$$[u(x_N) - U_N] + \sigma \left[D^{\alpha-\beta}_{C} u(x_N) - D^{\alpha-\beta}_{C,L1} U_N \right] = \sigma \left[D^{\alpha-\beta}_{C} u(x_N) - D^{\alpha-\beta}_{C,L1} U_N \right].$$

To establish the truncation error bound, we require the following lemma.

Lemma 4.1. *For all integers $j \geq 2$, there exists a constant C, independent of j, such that*

$$\sum_{k=1}^{j-1} \left[(j-k)^{1-\beta} - (j-k-1)^{1-\beta} \right] k^{\alpha-\beta-1} \leq C j^{\alpha-2\beta}. \tag{4.2}$$

Proof. For $x \in \mathbb{R}$, we apply the mean value theorem on the function $f(x) = x^{1-\beta}$ and proceed like [22, Lemma 4.4.] to get

$$\begin{aligned} \sum_{k=1}^{[j/2]-1} \left[(j-k)^{1-\beta} - (j-k-1)^{1-\beta} \right] k^{\alpha-\beta-1} &\leq \sum_{k=1}^{[j/2]-1} (1-\beta)(j-k-1)^{-\beta} k^{\alpha-\beta-1} \\ &\leq C j^{-\beta} \sum_{k=1}^{[j/2]-1} k^{\alpha-\beta-1}. \end{aligned}$$

Then, using the convergence concept given in [23, Eqn. (5.9)], we get

$$\sum_{k=1}^{[j/2]-1} \left[(j-k)^{1-\beta} - (j-k-1)^{1-\beta} \right] k^{\alpha-\beta-1} \leq C j^{-2} j^{\alpha-2\beta+2} = C j^{\alpha-2\beta}, \tag{4.3}$$

and for the remaining terms

$$\begin{aligned} \sum_{k=[j/2]}^{j-1} \left[(j-k)^{1-\beta} - (j-k-1)^{1-\beta} \right] k^{\alpha-\beta-1} &\leq \left[\frac{j}{2} \right]^{\alpha-\beta-1} \sum_{k=[j/2]}^{j-1} \left[(j-k)^{1-\beta} - (j-k-1)^{1-\beta} \right] \\ &\leq C j^{\alpha-2\beta}. \end{aligned} \tag{4.4}$$

Thus, combining (4.3) and (4.4) we get the required result (4.2). □

4.1.1. *Truncation error bound.*

Lemma 4.2. *Let $a, r, f \in \mathcal{C}^{k,\eta}(0,1]$ for some positive integer k with $\eta \leq 2\beta - \alpha$. Then, we have the following bound for the truncation errors:*

$$(i)|D^+(u(x_0) - U_0)| \leq Ch^{\alpha-\beta}, \tag{4.5}$$

$$(ii)\ \left|[u(x_N) - U_N] + \sigma\left[D_C^{\alpha-\beta}u(x_N) - D_{C,L1}^{\alpha-\beta}U_N\right]\right| \leq C\sigma h^{\min\{1+\alpha-\beta, 2-\alpha+\beta\}} \leq C\sigma h, \tag{4.6}$$

$$(iii)|\,L_N(u(x_j) - U_j)\,| \leq Chx_j^{-1}, \quad \text{for} \quad j = 1, 2, \ldots, N-1. \tag{4.7}$$

Proof. Our assumption clears that the FDE (1.1) has a unique solution.

(i) From [8, Lemma 4.1], using (2.6), we have

$$\begin{aligned}|D^+(u(x_0) - U_0)| &\leq \frac{1}{h}\left[\int_0^h \left(\int_0^t |u''(s)|\,ds\right)dt\right] \\ &\leq Ch^{-1}\int_0^h t^{\alpha-\beta}\,dt \\ &= Ch^{\alpha-\beta}.\end{aligned} \tag{4.8}$$

Thus, $|D^+(u(x_0) - U_0)| \leq Ch^{\alpha-\beta}$.

(ii) One can write this truncation error term as

$$\begin{aligned}D_C^{\alpha-\beta}u(x_N) - D_{C,L1}^{\alpha-\beta}U_N &= \frac{1}{\Gamma(1-\alpha+\beta)}\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}(x_N - s)^{\beta-\alpha}\left[u'(s) - \frac{U_{k+1} - U_k}{h}\right]ds \\ &= \sum_{k=0}^{N-1} t_{N,k},\end{aligned}$$

where

$$t_{N,k} = \frac{1}{\Gamma(1-\alpha+\beta)}\int_{x_k}^{x_{k+1}}(x_N - s)^{\beta-\alpha}\left[u'(s) - \frac{U_{k+1} - U_k}{h}\right]ds.$$

According to the derivation given in [23] and using (2.6), one gets

$$|t_{N,0}| \leq Ch^{1+\alpha-\beta}, \quad \text{and} \quad |t_{N,N-1}| \leq Ch^{2-\alpha+\beta}. \tag{4.9}$$

Now, for $k = 1, 2, \ldots, N-2$, and through (2.6), we emulate the derivation in [9, 23] to get

$$\begin{aligned}|t_{N,k}| &\leq Ch^2\left(\max_{s\in[x_k,x_{k+1}]}|u''(s)|\right)\int_{x_k}^{x_{k+1}}(x_N - s)^{\beta-\alpha-1}\,ds \\ &\leq Chk^{\alpha-\beta-1}\left[N - (k+1)\right]^{\beta-\alpha-1}.\end{aligned} \tag{4.10}$$

Inviting the bounds, given in [9, 23] for our problem, and using (2.6), we get

$$\sum_{k=1}^{[N/2]-1}|t_{N,k}| \leq Ch^2, \quad \text{and} \quad \sum_{k=[N/2]}^{N-2}|t_{N,k}| \leq Ch^{2+\beta-\alpha}. \tag{4.11}$$

Then, calling these bounds all together to get

$$\sigma\left|\left[D_C^{\alpha-\beta}u(x_N) - D_{C,L1}^{\alpha-\beta}U_N\right]\right| \leq C\sigma h^{\min\{1+\alpha-\beta, 2-\alpha+\beta\}} \leq C\sigma h. \tag{4.12}$$

(iii) It remains to find the bound for $x \in \Omega$. Following the idea given in [8, Lemma 4.1], we disintegrate the truncation error term for the mesh points $x_1, x_2, \ldots, x_{N-1}$ into three parts as follows:

$$\begin{aligned} L_N(u(x_j) - U_j) &= -\left(D^+ D^{\alpha-\beta}_{C,L1} - D(D^{\alpha-\beta}_C)\right) u(x_j) + a_j(D^{\beta}_{C,L1} - D^{\beta}_C) u(x_j) \\ &= \left(D - D^+\right) D^{\alpha-\beta}_C u(x_j) + D^+ \left(D^{\alpha-\beta}_C - D^{\alpha-\beta}_{C,L1}\right) u(x_j) + a_j \left(D^{\beta}_{C,L1} - D^{\beta}_C\right) u(x_j). \end{aligned} \tag{4.13}$$

For the first two parts in (4.13), one can obtain the following relations with the similar derivation of [8, Lemma 4.1] along with (2.6):

$$\left| \left(\frac{d}{dx} - D^+\right) D^{\alpha-\beta}_C u(x_j) \right| \le C h x_j^{\alpha-\beta-1}, \tag{4.14}$$

and

$$\left| D^+ \left(D^{\alpha-\beta}_C - D^{\alpha-\beta}_{C,L1}\right) u(x_j) \right| \le C h x_j^{-1}. \tag{4.15}$$

For the third part in (4.13), we follow the derivation of [22, Lemma 4.5] and get

$$\begin{aligned} \left(D^{\beta}_C - D^{\beta}_{C,L1}\right) u(x_j) &= \frac{1}{\Gamma(1-\beta)} \int_0^{x_j} (x_j - s)^{-\beta} u'(s)\, ds - \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^{j-1} \int_{x_k}^{x_{k+1}} \frac{U_{k+1} - U_k}{h} (x_j - s)^{-\beta}\, ds \\ &= \tau_{j,0} + \sum_{k=1}^{j-1} \tau_{j,k}. \end{aligned} \tag{4.16}$$

For $j = 2, 3, \ldots, N-1$, using the mean value theorem we get

$$\begin{aligned} \tau_{j,k} &= \frac{1}{\Gamma(1-\beta)} \int_{x_k}^{x_{k+1}} (x_j - s)^{-\beta} u'(s)\, ds - \frac{U_{k+1} - U_k}{h^{\beta}\Gamma(2-\beta)}\, d_{j-k} \\ &= \frac{h^{1-\beta}}{\Gamma(2-\beta)} \left[u'(\eta) - \frac{U_{k+1} - U_k}{h}\right] d_{j-k}, \quad \eta \in (x_k, x_{k+1}). \end{aligned}$$

Thus,

$$\left| u'(\eta) - \frac{U_{k+1} - U_k}{h} \right| \le C h x_k^{\alpha-\beta-1}. \tag{4.17}$$

Hence, we have

$$|\, \tau_{j,k} \,| \le \frac{C h^{1+\alpha-2\beta}}{\Gamma(2-\beta)} k^{\alpha-\beta-1} d_{j-k}. \tag{4.18}$$

Then,

$$\left| \sum_{k=1}^{j-1} \tau_{j,k} \right| \le \frac{C h^{1+\alpha-2\beta}}{\Gamma(2-\beta)} \sum_{k=1}^{j-1} d_{j-k} k^{\alpha-\beta-1}.$$

Using Lemma 4.1, we get

$$\left| \sum_{k=1}^{j-1} \tau_{j,k} \right| \le \frac{C h^{1+\alpha-2\beta}}{\Gamma(2-\beta)} j^{\alpha-2\beta} \le C h x_j^{-(2\beta-\alpha)}. \tag{4.19}$$

We have

$$\left| \left(D^{\beta}_C - D^{\beta}_{C,L1}\right) u(x_j) \right| \le C h x_j^{-(2\beta-\alpha)}, \quad j = 2, 3, \ldots, N-1. \tag{4.20}$$

For $k = 0$,

$$\tau_{j,0} = \frac{1}{\Gamma(1-\beta)} \int_{x_0}^{x_1} (x_j - s)^{-\beta} u'(s)\, ds - \frac{U_1 - U_0}{h^{\beta}\Gamma(2-\beta)}\, d_j.$$

Following accordingly to the derivation of [22, Lemma 4.5], one gets

$$\left|\frac{1}{\Gamma(1-\beta)}\int_{x_0}^{x_1}(x_j-s)^{-\beta}u'(s)\,ds\right| \le Chx_{j-1}^{-\beta}, \tag{4.21}$$

and

$$\left|\frac{U_1-U_0}{h^\beta\Gamma(2-\beta)}\,d_j\right| \le Chx_{j-1}^{-\beta}, \tag{4.22}$$

using the mean value theorem.

Combining (4.21) and(4.22) and using $x_j \le 2x_{j-1}$ yields

$$\begin{aligned}|\,\tau_{j,0}\,| &\le Cx_{j-1}^{-\beta}h^{\alpha-\beta}\\ &\le Chx_j^{-\beta}, \quad j=2,3,\ldots,N-1.\end{aligned} \tag{4.23}$$

Finally, the bound for $j=1$ will complete the bound of $\tau_{j,k}$.

$$|\,\tau_{1,0}\,| \le \left|\frac{1}{\Gamma(1-\beta)}\int_{x_0}^{x_1}(x_1-s)^{-\beta}u'(s)\,ds\right| + \left|\frac{U_1-U_0}{h^\beta\Gamma(2-\beta)}\,d_1\right|.$$

Simulating the derivation of [22, Lemma 4.5], we get

$$\begin{aligned}\left|\frac{1}{\Gamma(1-\beta)}\int_{x_0}^{x_1}(x_1-s)^{-\beta}u'(s)\,ds\right| &\le \frac{C}{\Gamma(1-\beta)}\int_{x_0}^{x_1}(x_1-s)^{-\beta}s^{\alpha-\beta}\,ds\\ &\le Ch{x_1}^{-(2\beta-\alpha)},\end{aligned} \tag{4.24}$$

and

$$\left|\frac{d_1}{h^\beta\Gamma(2-\beta)}\,(U_1-U_0)\right| \le \frac{Cd_1h^{\alpha-\beta}}{h^\beta\Gamma(2-\beta)}h \le Ch{x_1}^{-(2\beta-\alpha)}. \tag{4.25}$$

Thus, (4.24) and (4.25) together yield

$$|\,\tau_{1,0}\,| \le Ch{x_1}^{-(2\beta-\alpha)}. \tag{4.26}$$

Putting all these inequalities (4.20)-(4.26) together, we obtain

$$\left|\left(D_C^\beta - D_{C,L1}^\beta\right)u(x_j)\right| \le Chx_j^{-\beta} \le Chx_j^{-1}, \quad j=1,2,\ldots,N-1. \tag{4.27}$$

We get the required bound (4.7) by adding the bounds (4.14)-(4.27). □

The following theorem provides the bound of the error in discrete maximum norm.

Theorem 4.3. *Let the solution $u(x)$ of the problem (1.1) satisfy the discrete comparison principle (Lemma 3.2) as well as the truncation error bounds (4.5)–(4.7). Also, assume the coefficients $a(x)$ and $r(x)$ are chosen such that $a(x) \le 0$ and $r(x) \ge 0$ for $x \in \overline{\Omega}$. Then, there exists a constant C such that*

$$\max_{0\le i\le N}|e_i| = \max_{0\le i\le N}|u(x_i)-U_i| \le Ch|\ln h|^\beta.$$

Before proceeding towards the proof of the above theorem, we shall enter into the discussion of the discrete barrier function that will be used later in the proof of the above theorem. We first consider a non-negative mesh function $\{\Theta_i\}_{i=0}^N$, which satisfies

$$|L_N(u(x_i)-U_i)| \le L_N\Theta_i, \quad \text{for } i=1,2,\ldots,N-1, \tag{4.28}$$

$$|-D^+(u(x_0)-U_0)| \le -D^+\Theta_0, \tag{4.29}$$

$$|(u(x_N)-U_N) + \sigma\left(D_C^{\alpha-\beta}u(x_N) - D_{C,L1}^{\alpha-\beta}U_N\right)| \le \Theta_N + \sigma D_{C,L1}^{\alpha-\beta}\Theta_N. \tag{4.30}$$

Then, use of the discrete comparison principle over $\Theta_i \pm (u(x_i) - U_i)$ gives

$$|u(x_i) - U_i| \le \Theta_i, \quad \text{for all} \quad i \ge 0.$$

This mesh function $\{\Theta_i\}_{i=0}^N$ is called the discrete barrier function.

Now, we call a lemma from [8, Lemma 4.2], which will be useful for the study of barrier function and error estimate. The following lemma is given in the suitable form for our problem.

Lemma 4.4. *For a mesh function $\{G_i\}_{i=0}^N$ along with the condition $G_0 = 0$ and $G_i \le G_{i+1}$, one has*

$$D_{C,L1}^{\alpha-\beta} G_i \ge \frac{G_i}{{x_i}^{\alpha-\beta}\Gamma(1-\alpha+\beta)}, \quad \textit{for} \quad 1 \le i \le N-1.$$

Proof. For the proof, one can follow [8, Lemma 4.2]. □

We next define a non-negative mesh function [8, Eq. 4.11], $\{M_i\}_{i=0}^N$ for a smooth construction of the suitable discrete barrier function for our problem, by

$$D_{C,L1}^{\alpha-\beta} M_i = \frac{e^{|\ln h|^{1-\beta}}}{\Gamma(1-\alpha+\beta)} {x_i}^{|\ln h|^{-\beta}}, \quad i = 1, 2, \ldots, N, \quad \text{and} \quad M_0 = 0. \tag{4.31}$$

Following the derivation given in [8, Section 4.2], one has

(1) $M_1 = (1-\alpha+\beta)h^{\alpha-\beta}$,
(2) M_i is a non-decreasing function for $i \ge 0$,
(3) $0 \le M_i \le H_i$, $i = 0, 1, \ldots, N$, where the mesh function H_i plays the role of discrete barrier function for M_i, defined by

$$H_i = {x_i}^{\alpha-\beta+|\ln h|^{-\beta}} e^{|\ln h|^{1-\beta}}.$$

We now begin the proof of Theorem 4.3.

Proof. We start the proof with the consideration of two cases:

$$\sigma = 0, \quad \text{and} \quad \sigma > 0\,.$$

Case-1 ($\sigma = 0$). Define a mesh function with the help of the mesh function M_i, defined in (4.31), by

$$\Theta_i = C_1\, h\, |\ln h|^\beta \left(e^{|\ln h|^{1-\beta}} - M_i\right), \quad \text{for} \quad i = 0, 1, \ldots, N. \tag{4.32}$$

Clearly one can obtain $0 \le \Theta_i \le C\, h\, |\ln h|^\beta$. Our aim is to show that Θ_i is a discrete barrier function of the error function e_i.

From the upper bound of M_i, it can be said that Θ_i attains the non-negative values for all i.

Proceeding similarly as given in [8, Theorem 4.1], we have at the left endpoint $x = 0$,

$$-D^+\Theta_0 = C_1(1-\alpha+\beta)h^{\alpha-\beta}|\ln h|^\beta.$$

As the mesh function M_i is non-decreasing and $r(x) \ge 0$, using the non-positive condition over the function $a(x)$, we have

$$a_i\, D_{C,L1}^\beta \Theta_i + c_i\Theta_i = -a_i\, C_1\, h\, |\ln h|^\beta D_{C,L1}^\beta M_i + c_i\Theta_i \ge 0\,.$$

Then for x_j, $j = 1, 2, \ldots, N-1$, in the similar way given in [8], we have

$$\begin{aligned} L_N\Theta_i &:= -D^+(D_{C,L1}^{\alpha-\beta}\Theta_i) + a_i D_{C,L1}^\beta\Theta_i + c_i\Theta_i \\ &\ge C_1\, h\, |\ln h|^\beta D^+(D_{C,L1}^{\alpha-\beta} M_i) \\ &\ge \frac{C_1\, h}{2\Gamma(1-\alpha+\beta)}\, x_i^{-1}. \end{aligned} \tag{4.33}$$

Also, $u(x_N) - U_N = 0 \le \Theta_N$.

Then, the combination of the above inequalities along with the discrete comparison principle implies that Θ_i is a discrete barrier function for the error $|u(x_i) - U_i|$, and hence the result follows for the case $\sigma = 0$.

Case-2 ($\sigma > 0$) At the point x_N,

$$\begin{aligned}\Theta_N + \sigma\, D^{\alpha-\beta}_{C,L1}\Theta_N \;&=\; \Theta_N - C_1\sigma h|\ln h|^{\beta}\frac{e^{|\ln h|^{1-\beta}}}{\Gamma(1-\alpha+\beta)}\\ &\geq \frac{-C_1\sigma h|\ln h|^{\beta}e^{|\ln h|^{1-\beta}}}{\Gamma(1-\alpha+\beta)}.\end{aligned} \tag{4.34}$$

Thus, we can see that Θ_i is not the perfect choice for the case $\sigma > 0$, and hence some changes are required in the structure of the discrete barrier function.

We define the modified mesh function as

$$\widetilde{\Theta}_i = \Theta_i + \Upsilon_i,$$

where the construction of Υ_i is followed from [8] and defined by

$$\Upsilon_i = \begin{cases} C_2 h|\ln h|^{\beta}\left(\dfrac{h^{\alpha-\beta}}{\sigma} + \dfrac{2}{\Gamma(2-\alpha+\beta)}\right)e^{|\ln h|^{1-\beta}}, & \text{for } i = 0,1,\dots,N-1,\\[2ex] C_2 h|\ln h|^{\beta}\dfrac{2e^{|\ln h|^{1-\beta}}}{\Gamma(2-\alpha+\beta)}, & \text{for} \quad i = N. \end{cases} \tag{4.35}$$

At the point x_0,

$$-D^+\widetilde{\Theta}_0 = -D^+\Theta_0 - D^+\Upsilon_0 \geq C_1C_3\Gamma(2-\alpha+\beta)h^{\alpha-\beta}|\ln h|^{\beta}.$$

It is easy to calculate the following:

(1) $D^{\alpha-\beta}_{C,L1}\Upsilon_i = 0, \quad \text{for} \quad i = 1,2,\dots,N-1,$

(2) $D^{+}(D^{\alpha-\beta}_{C,L1}\Upsilon_i) = 0, \quad \text{for} \quad i = 1,2,\dots,N-2,$

(3) $D^{\beta}_{C,L1}\Upsilon_i = 0, \quad \text{for} \quad i = 1,2,\dots,N-1.$

But at the point x_N,

$$\begin{aligned} D^{\alpha-\beta}_{C,L1}\Upsilon_N &= \frac{1}{h^{\alpha-\beta}\Gamma(2-\alpha+\beta)}\sum_{k=0}^{N-1}(\Upsilon_{k+1}-\Upsilon_k)b_{N-k}\\ &= \frac{\Upsilon_N - \Upsilon_{N-1}}{h^{\alpha-\beta}\Gamma(2-\alpha+\beta)}b_1\\ &= -\frac{C_2h|\ln h|^{\beta}h^{\alpha-\beta}}{\sigma h^{\alpha-\beta}\Gamma(2-\alpha+\beta)}e^{|\ln h|^{1-\beta}}\\ &= -\frac{C_2h|\ln h|^{\beta}e^{|\ln h|^{1-\beta}}}{\sigma\Gamma(2-\alpha+\beta)} \leq 0. \end{aligned}$$

Similarly, one can have

$$D^{\beta}_{C,L1}\Upsilon_N = -\frac{C_2h^{1+\alpha-2\beta}|\ln h|^{\beta}}{\sigma\Gamma(2-\beta)}e^{|\ln h|^{1-\beta}} \leq 0.$$

Therefore,

$$-D^{+}(D^{\alpha-\beta}_{C,L1}\Upsilon_{N-1}) \;\geq\; 0.$$

We next calculate $L_N\widetilde{\Theta}_i$, for $1 \leq i \leq N-1$.

For $1 \le i \le N-2$,

$$L_N\widetilde{\Theta}_i = L_N\Theta_i + L_N\Upsilon_i = L_N\Theta_i + c_i\Upsilon_i \ge \frac{C_1\, h}{2\Gamma(1-\alpha+\beta)}\, x_i^{-1}.$$

At $x = x_{N-1}$,

$$L_N\widetilde{\Theta}_{N-1} = L_N\Theta_{N-1} + L_N\Upsilon_{N-1} \ge L_N\Theta_{N-1} + c_{N-1}\Theta_{N-1}.$$

Inserting the condition $r(x) \ge 0$, we get

$$\begin{aligned} L_N\widetilde{\Theta}_{N-1} &\ge L_N\Theta_{N-1} \\ &\ge \frac{C_1\, h}{2\Gamma(1-\alpha+\beta)}\, x_{N-1}^{-1}. \end{aligned} \tag{4.36}$$

Now,

$$\begin{aligned} \Upsilon_N + \sigma\, D_{C,L1}^{\alpha-\beta}\Upsilon_N \; &= \; C_2 h|\ln\, h|^{\beta}\frac{2e^{|\ln\, h|^{1-\beta}}}{\Gamma(2-\alpha+\beta)} - \frac{\sigma C_2 h|\ln\, h|^{\beta} e^{|\ln\, h|^{1-\beta}}}{\sigma\Gamma(2-\alpha+\beta)} \\ &= C_2 h|\ln\, h|^{\beta}\frac{e^{|\ln\, h|^{1-\beta}}}{\Gamma(2-\alpha+\beta)}. \end{aligned} \tag{4.37}$$

Finally, at $x = x_N$,

$$\begin{aligned} \widetilde{\Theta}_N + \sigma\, D_{C,L1}^{\alpha-\beta}\widetilde{\Theta}_N \; &= \Big(\Theta_N + \sigma\, D_{C,L1}^{\alpha-\beta}\Theta_N\Big) + \Big(\Upsilon_N + \sigma\, D_{C,L1}^{\alpha-\beta}\Upsilon_N\Big) \\ &\ge h|\ln\, h|^{\beta}\left[\frac{C_2}{\Gamma(2-\alpha+\beta)} - \frac{C_1\sigma}{\Gamma(1-\alpha+\beta)}\right] e^{|\ln\, h|^{1-\beta}} \\ &\ge Ch|\ln\, h|^{\beta}, \quad \text{as} \quad e^{|\ln\, h|^{1-\beta}} \ge 1, \end{aligned} \tag{4.38}$$

where C_2 is chosen in such a way that $\dfrac{C_2}{\Gamma(2-\alpha+\beta)} - \dfrac{C_1\sigma}{\Gamma(1-\alpha+\beta)} > C$.

Thus, $\widetilde{\Theta}_i$ is the discrete barrier function of a $|u(x_i) - U_i|$ in case of positive value of σ, and hence the result follows. □

5. Semilinear Fractional Differential Equation and Quasilinearization Technique

In this section, we demonstrate the application of the proposed scheme to a semilinear FDE.

Let us consider the following semilinear FDE:

$$\begin{cases} -D(D_C^{\alpha-\beta}v(x)) + a(x)D_C^{\beta}v(x) + \mathcal{G}(x, v(x)) = 0, \quad x \in \Omega = (0,1), \\ D_C^{\alpha-\beta}v(0) = 0, \quad v(1) + \sigma D_C^{\alpha-\beta}v(1) = \gamma. \end{cases} \tag{5.1}$$

Assuming the functions $a(x)$ and $\mathcal{G}(x, v(x))$ are sufficiently smooth, the FDE (5.1) generally has a unique solution $v(x)$.

To numerically solve (5.1), we employ the Newton's method of quasilinearization technique. This method generates a sequence $\left\{v^{(q)}\right\}_0^{\infty}$ as the solution with a suitable initial guess $v^{(0)}(x)$ for $q \ge 0$.

For each fixed non-negative integer q, we define $v^{(q+1)}$ to be the solution of the following linear FDE:

$$\begin{cases} -D(D_C^{\alpha-\beta}v^{(q+1)}(x)) + a(x)D_C^{\beta}v^{(q+1)}(x) + \mathfrak{r}^{(q)}(x)v^{(q+1)}(x) = \mathcal{F}^{(q)}(x), \quad x \in \Omega, \\ D_C^{\alpha-\beta}v^{(q+1)}(0) = 0, \quad v^{(q+1)}(1) + \sigma D_C^{\alpha-\beta}v^{(q+1)}(1) = \gamma, \end{cases} \tag{5.2}$$

where $\mathfrak{r}^{(q)}(x)$ and $\mathcal{F}^{(q)}(x)$ are given by

$$\begin{cases} \mathfrak{r}^{(q)}(x) = \dfrac{\partial g}{\partial v}(x, v^{(q)}), \\ \mathcal{F}^{(q)}(x) = \mathfrak{r}^{(q)}(x)v^{(q)} - \mathcal{G}(x, v^{(q)}). \end{cases} \tag{5.3}$$

Therefore, for each fixed q, we utilize the proposed numerical scheme (3.1) to solve (5.2). To facilitate the Newton's quasilinearization process, we will implement this convergence criterion:

$$|v^{(q+1)}(x_i) - v^{(q)}(x_i)| \leq tol, \quad x_i \in \overline{\Omega}, \quad q \geq 0. \tag{5.4}$$

For computational purposes, we set $tol = 10^{-7}$.

6. Numerical Experiments

In this section, numerical examples are presented to affirm the theoretical results. The main motive is to check the error estimates and convergence rates for different values of α and β.

Example 6.1. Consider the following FDE with constant coefficient:

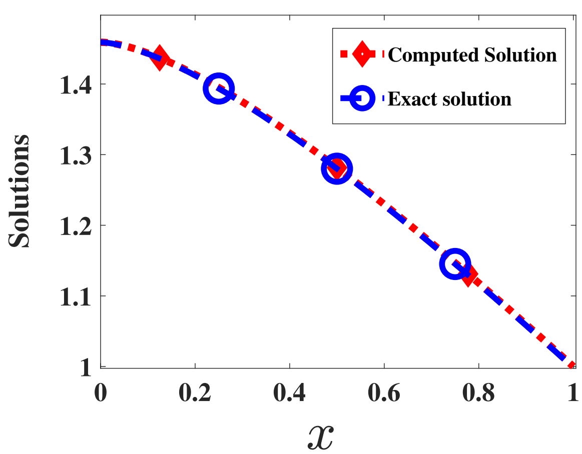
$$\begin{cases} -D(D_C^{\alpha-\beta}u(x)) - 0.5D_C^{\beta}u(x) = 1, \quad 0 < \alpha - \beta < \beta \leq 1 < \alpha < 2, \quad x \in \Omega, \\ D_C^{\alpha-\beta}u(0) = 0, \quad u(1) = 0. \end{cases} \tag{6.1}$$

The exact solution can be found by using (2.4).

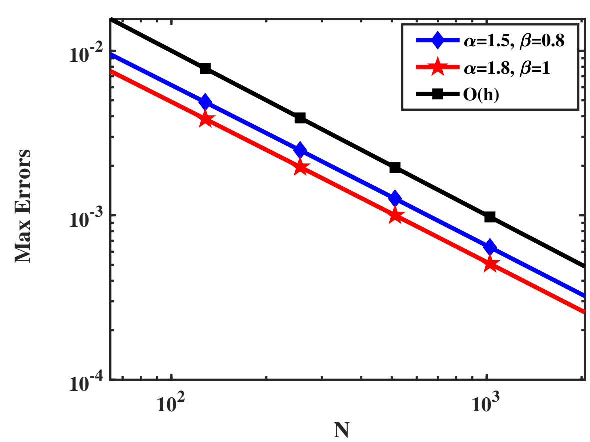
The maximum error and the associated convergence orders are determined by

$$M_N = \max_{0\leq j\leq N} |U_j - u(x_j)|, \quad \text{and} \quad C_N = \log_2\left(\frac{M_N}{M_{2N}}\right),$$

respectively.



(a) Solutions for $\alpha = 1.4$, $\beta = 0.8$, and $N = 256$.



(b) Log-log plots for various values of α and β.

Figure 1. Figures corresponding to Example 6.1.

The maximum pointwise error and the corresponding first-order convergence of the numerical results of Example 6.1 are presented in Table 1 for $\beta = 0.8$. The results, presented in Table 2, demonstrate that Equation 6.1 yields better results than those in [8, Table 1] for $\beta = 1$. Additionally, for $\beta = 0.8$ and $\alpha = 1.4$, Figure 1(a) illustrates the comparison between the exact and computed solutions with $N = 256$, while the log-log plot is shown in Figure 1(b).

TABLE 1. Maximum errors and orders of convergence for $\beta = 0.8$ of Example 6.1.

$\beta = 0.8$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	8.7367e-03	4.4137e-03	2.2265e-03	1.1218e-03	5.6459e-04	2.8390e-04
C_N	0.9851	0.9872	0.9890	0.9905	0.9918	
$\alpha = 1.2$	9.0831e-03	4.5940e-03	2.3193e-03	1.1691e-03	5.8862e-04	2.9605e-04
C_N	0.9834	0.9861	0.9882	0.9900	0.9915	
$\alpha = 1.3$	9.3833e-03	4.7562e-03	2.4050e-03	1.2137e-03	6.1157e-04	3.0776e-04
C_N	0.9803	0.9838	0.9866	0.9889	0.9907	
$\alpha = 1.4$	9.5678e-03	4.8683e-03	2.4691e-03	1.2490e-03	6.3049e-04	3.1773e-04
C_N	0.9748	0.9794	0.9832	0.9862	0.9887	
$\alpha = 1.5$	9.5215e-03	4.8724e-03	2.4833e-03	1.2614e-03	6.3898e-04	3.2297e-04
C_N	0.9666	0.9724	0.9772	0.9812	0.9844	

TABLE 2. Maximum errors and orders of convergence for $\beta = 1$ of Example 6.1.

$\beta = 1$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	7.4982e-03	3.5549e-03	1.7734e-03	8.8668e-04	4.4334e-04	2.2167e-04
C_N	1.0767	1.0033	1.0000	1.0000	1.0000	
$\alpha = 1.2$	7.2712e-03	3.6359e-03	1.8181e-03	9.0913e-04	4.5459e-04	2.2730e-04
C_N	0.9999	0.9999	0.9999	0.9999	1.0000	
$\alpha = 1.3$	7.4703e-03	3.7378e-03	1.8699e-03	9.3527e-04	4.6775e-04	2.3391e-04
C_N	0.9990	0.9993	0.9995	0.9996	0.9998	
$\alpha = 1.4$	7.6781e-03	3.8473e-03	1.9267e-03	9.6440e-04	4.8257e-04	2.4141e-04
C_N	0.9969	0.9977	0.9984	0.9989	0.9992	
$\alpha = 1.5$	7.8661e-03	3.9522e-03	1.9834e-03	9.9439e-04	4.9818e-04	2.4945e-04
C_N	0.9930	0.9947	0.9961	0.9971	0.9979	
$\alpha = 1.6$	7.9789e-03	4.0270e-03	2.0284e-03	1.0201e-03	5.1229e-04	2.5701e-04
C_N	0.9865	0.9893	0.9917	0.9936	0.9951	
$\alpha = 1.7$	7.9196e-03	4.0226e-03	2.0377e-03	1.0298e-03	5.1944e-04	2.6157e-04
C_N	0.9773	0.9812	0.9845	0.9874	0.9897	
$\alpha = 1.8$	7.5326e-03	3.8530e-03	1.9654e-03	9.9988e-04	5.0751e-04	2.5706e-04
C_N	0.9672	0.9712	0.9750	0.9783	0.9813	
$\alpha = 1.9$	6.5862e-03	3.3766e-03	1.7285e-03	8.8332e-04	4.5065e-04	2.2954e-04
C_N	0.9639	0.9660	0.9685	0.9709	0.9733	

Example 6.2. Consider the following FDE with variable coefficients:

$$\begin{cases} -D(D_C^{\alpha-\beta}u(x)) - (1+x^2)D_C^{\beta}u(x) + e^{-x}u(x) = x, \quad x \in \Omega, \\ D_C^{\alpha-\beta}u(0) = 0, \qquad u(1) + D_C^{\alpha-\beta}u(1) = 1, \end{cases} \tag{6.2}$$

where $0 < \alpha - \beta < \beta \leq 1 < \alpha < 2$.

The double-mesh principle [8] is used to derive the order of convergence as the exact solution is obscured to us. From the scheme (3.1), we obtain two solutions, $\{U_j\}_{j=0}^{N}$ and $\{\widetilde{U}_j\}_{j=0}^{2N}$, using the uniform meshes $\{x_j\}_{j=0}^{N}$ and $\{\tilde{x}_j\}_{j=0}^{2N}$, where $x_j = \tilde{x}_{2j}$ for $j = 0, 1, \ldots, N$.

The errors computed using the double-mesh principle, along with the corresponding orders of convergence, are evaluated by

$$E_N = \max_{0\le j\le N} |U_j - \widetilde{U}_{2j}|, \quad \text{and} \quad \widetilde{C}_N = \log_2\left(\frac{E_N}{E_{2N}}\right),$$

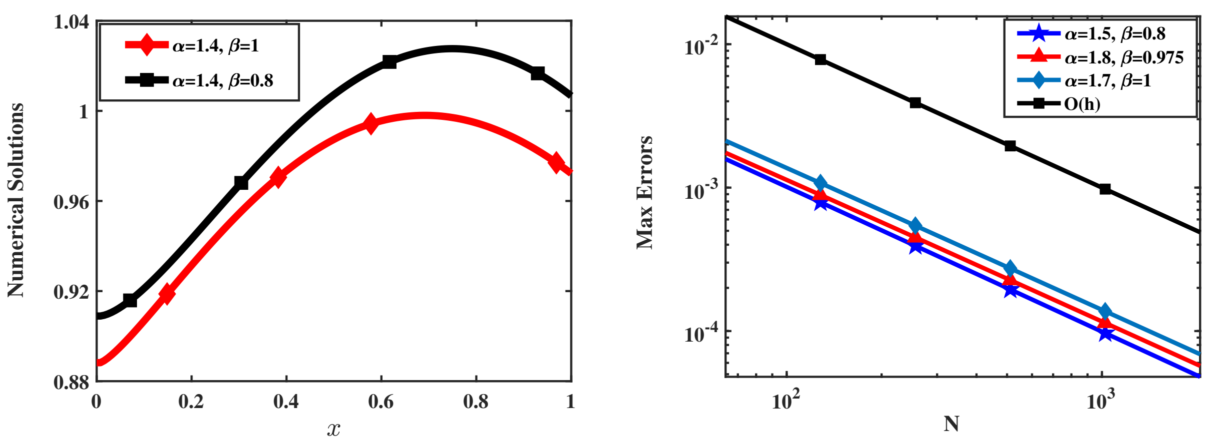
respectively.

Figure 2(a) depicts the numerical solutions for $\beta = 0.8$ and $\alpha = 1.4$, and $\beta = 1$ and $\alpha = 1.4$ with $N = 128$. Log-log plots are displayed for various values of α and β in Figure 2(b).

Tables 3–5 cover the maximum differences and orders of convergence obtained by using the double-mesh principle for $\beta = 0.8, 0.975$, and 1 respectively. The results in these tables demonstrate the first-order convergence.

TABLE 3. Maximum differences and orders of convergence obtained by the double-mesh principle for $\beta = 0.8$ of Example 6.2.

$\beta = 0.8$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	2.2327e-03	1.1024e-03	5.4419e-04	2.6879e-04	1.3290e-04	6.5778e-05
$\widetilde{C}_N$	1.0181	1.0185	1.0176	1.0162	1.0149	
$\alpha = 1.2$	2.1064e-03	1.0414e-03	5.1435e-04	2.5410e-04	1.2563e-04	6.2171e-05
$\widetilde{C}_N$	1.0163	1.0177	1.0174	1.0162	1.0148	
$\alpha = 1.3$	1.9601e-03	9.7124e-04	4.8031e-04	2.3745e-04	1.1744e-04	5.8129e-05
$\widetilde{C}_N$	1.0130	1.0159	1.0163	1.0157	1.0146	
$\alpha = 1.4$	1.7871e-03	8.8853e-04	4.4037e-04	2.1803e-04	1.0795e-04	5.3472e-05
$\widetilde{C}_N$	1.0081	1.0127	1.0142	1.0142	1.0135	
$\alpha = 1.5$	1.5829e-03	7.9021e-04	3.9279e-04	1.9492e-04	9.6691e-05	4.7972e-05
$\widetilde{C}_N$	1.0023	1.0085	1.0109	1.0114	1.0112	



(a) Numerical solutions of Example 6.2 for $N = 128$.

(b) Log-log plots for various values of α and β.

FIGURE 2. Figures corresponding to Example 6.2.

TABLE 4. Maximum differences and orders of convergence obtained by the double-mesh principle for $\beta = 0.975$ of Example 6.2.

$\beta = 0.975$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	2.8506e-03	1.4235e-03	7.1025e-04	3.5423e-04	1.7663e-04	8.8071e-05
$\widetilde{C}_N$	1.0018	1.0031	1.0037	1.0039	1.0040	
$\alpha = 1.2$	2.7425e-03	1.3701e-03	6.8373e-04	3.4103e-04	1.7006e-04	8.4796e-05
$\widetilde{C}_N$	1.0012	1.0028	1.0035	1.0038	1.0040	
$\alpha = 1.3$	2.6303e-03	1.3151e-03	6.5658e-04	3.2758e-04	1.6338e-04	8.1468e-05
$\widetilde{C}_N$	1.0001	1.0021	1.0031	1.0036	1.0039	
$\alpha = 1.4$	2.5091e-03	1.2563e-03	6.2781e-04	3.1340e-04	1.5636e-04	7.7986e-05
$\widetilde{C}_N$	0.99794	1.0008	1.0023	1.0031	1.0036	
$\alpha = 1.5$	2.3715e-03	1.1904e-03	5.9580e-04	2.9775e-04	1.4867e-04	7.4191e-05
$\widetilde{C}_N$	0.99440	0.99850	1.0007	1.0020	1.0028	
$\alpha = 1.6$	2.2073e-03	1.1120e-03	5.5806e-04	2.7944e-04	1.3974e-04	6.9816e-05
$\widetilde{C}_N$	0.98908	0.99469	0.99789	0.99983	1.0011	
$\alpha = 1.7$	2.0039e-03	1.0143e-03	5.1089e-04	2.5658e-04	1.2863e-04	6.4403e-05
$\widetilde{C}_N$	0.98227	0.98942	0.99360	0.99623	0.9980	
$\alpha = 1.8$	1.7497e-03	8.8937e-04	4.4956e-04	2.2651e-04	1.1389e-04	5.7189e-05
$\widetilde{C}_N$	0.97623	0.98427	0.98894	0.99188	0.9939	
$\alpha = 1.9$	1.4399e-03	7.3111e-04	3.6933e-04	1.8606e-04	9.3583e-05	4.7025e-05
$\widetilde{C}_N$	0.97778	0.98518	0.98915	0.99143	0.9928	

TABLE 5. Maximum differences and orders of convergence obtained by the double-mesh principle for $\beta = 1$ of Example 6.2.

$\beta = 1$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	2.9408e-03	1.4726e-03	7.3687e-04	3.6857e-04	1.8432e-04	9.2167e-05
$\widetilde{C}_N$	0.9978	0.9989	0.9995	0.9997	0.9999	
$\alpha = 1.2$	2.8313e-03	1.4183e-03	7.0979e-04	3.5506e-04	1.7757e-04	8.8795e-05
$\widetilde{C}_N$	0.9973	0.9987	0.9993	0.9997	0.9998	
$\alpha = 1.3$	2.7193e-03	1.3631e-03	6.8244e-04	3.4145e-04	1.7078e-04	8.5409e-05
$\widetilde{C}_N$	0.9963	0.9981	0.9990	0.9995	0.9997	
$\alpha = 1.4$	2.6006e-03	1.3052e-03	6.5393e-04	3.2734e-04	1.6378e-04	8.1922e-05
$\widetilde{C}_N$	0.9946	0.9971	0.9983	0.9990	0.9994	
$\alpha = 1.5$	2.4684e-03	1.2415e-03	6.2290e-04	3.1210e-04	1.5626e-04	7.8197e-05
$\widetilde{C}_N$	0.9915	0.9950	0.9970	0.9981	0.9988	
$\alpha = 1.6$	2.3134e-03	1.1674e-03	5.8708e-04	2.9467e-04	1.4772e-04	7.3999e-05
$\widetilde{C}_N$	0.9867	0.9917	0.9945	0.9962	0.9973	
$\alpha = 1.7$	2.1228e-03	1.0761e-03	5.4302e-04	2.7330e-04	1.3732e-04	6.8917e-05
$\widetilde{C}_N$	0.9802	0.9867	0.9905	0.9929	0.9946	
$\alpha = 1.8$	1.8832e-03	9.5908e-04	4.8587e-04	2.4537e-04	1.2366e-04	6.2239e-05
$\widetilde{C}_N$	0.9735	0.9811	0.9856	0.9885	0.9905	
$\alpha = 1.9$	1.5851e-03	8.0834e-04	4.1008e-04	2.0743e-04	1.0474e-04	5.2820e-05
$\widetilde{C}_N$	0.9715	0.9791	0.9833	0.9859	0.9876	

Example 6.3. Let us consider the following semilinear FDE:

$$\begin{cases} -D(D_C^{\alpha-\beta}u(x)) - (1+x/2)D_C^{\beta}u(x) + \mathcal{G}(x,u(x)) = 0, \quad x \in \Omega, \\ D_C^{\alpha-\beta}u(0) = 0, \qquad u(1) + 0.5\, D_C^{\alpha-\beta}u(1) = \gamma, \end{cases} \tag{6.3}$$

where $\gamma = 3 + 1/2\,(\Gamma(2+\alpha-\beta) + 1/\Gamma(2-\alpha+\beta))$ and the nonlinear function $\mathcal{G}(x,u(x))$ is given by

$$\mathcal{G}(x,u(x)) = 2u(x) + u^2(x) + \eta(x).$$

The function $\eta(x)$ is selected so that the exact solution of the FDE (6.3) is $u(x) = x^{\alpha-\beta+1} \in \mathcal{C}^1(\overline{\Omega})$.

Now, utilizing the quasilinearization technique given in (5.2), we derive the following sequence of linear FDEs:

$$\begin{cases} -D(D_C^{\alpha-\beta}u^{(q+1)}(x)) - (1+x/2)D_C^{\beta}u^{(q+1)}(x) + (2u^{(q)}+2)u^{(q+1)}(x) \\ = (2u^{(q)}+2)u^{(q)}(x) - \mathcal{G}(x,u^{(q)}), \quad x \in \Omega, \\ D_C^{\alpha-\beta}u^{(q+1)}(0) = 0, \qquad u^{(q+1)}(1) + 0.5D_C^{\alpha-\beta}u^{(q+1)}(1) = \gamma. \end{cases} \tag{6.4}$$

Then, for fixed q, we solve (6.4) using the discretization method discussed earlier. After reaching the tolerance bound mentioned in (5.4), we break the Newton sequence and consider that as the solution of our problem.

The maximum pointwise error and its associated order of convergence are determined using the same approach as illustrated in Example 6.1. The experimental outcomes of Example 6.3, presented in Tables 6-7, show the good agreement with the theoretical estimates. Log-log plots of Example 6.3 with $\alpha = 1.5, \beta = 0.8$, and $\alpha = 1.7, \beta = 1$ are portrayed in Figure 3(b). Additionally, Figure 3(a) offers a direct visual comparison of the exact solution and the approximate solution for the case when $N = 256$ and $\alpha = 1.5$, illustrating the accuracy of the numerical method for this specific parameter choice.

Table 6. Maximum errors and orders of convergence obtained for $\beta = 0.8$ of Example 6.3.

$\beta = 0.8$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	8.3169e-03	4.2106e-03	2.1331e-03	1.0802e-03	5.4645e-04	2.7615e-04
C_N	0.9820	0.9811	0.9817	0.9831	0.9847	
$\alpha = 1.2$	8.5603e-03	4.3577e-03	2.2167e-03	1.1259e-03	5.7081e-04	2.8889e-04
C_N	0.9741	0.9751	0.9774	0.9800	0.9825	
$\alpha = 1.3$	8.8741e-03	4.5415e-03	2.3193e-03	1.1813e-03	6.0012e-04	3.0416e-04
C_N	0.9664	0.9695	0.9733	0.9771	0.9804	
$\alpha = 1.4$	9.1986e-03	4.7332e-03	2.4276e-03	1.2406e-03	6.3185e-04	3.2086e-04
C_N	0.9586	0.9633	0.9685	0.9734	0.9776	
$\alpha = 1.5$	9.4517e-03	4.8906e-03	2.5205e-03	1.2935e-03	6.6114e-04	3.3677e-04
C_N	0.9506	0.9563	0.9625	0.9682	0.9732	

6.1. **Experimental results for the case $\alpha - \beta = 1$.** In this subsection, we present some numerical results of Examples 6.1–6.3 for various values of α, $1 < \alpha \leq 2$, and β, $0 < \beta \leq 1$, that satisfy the condition $\alpha - \beta = 1$.

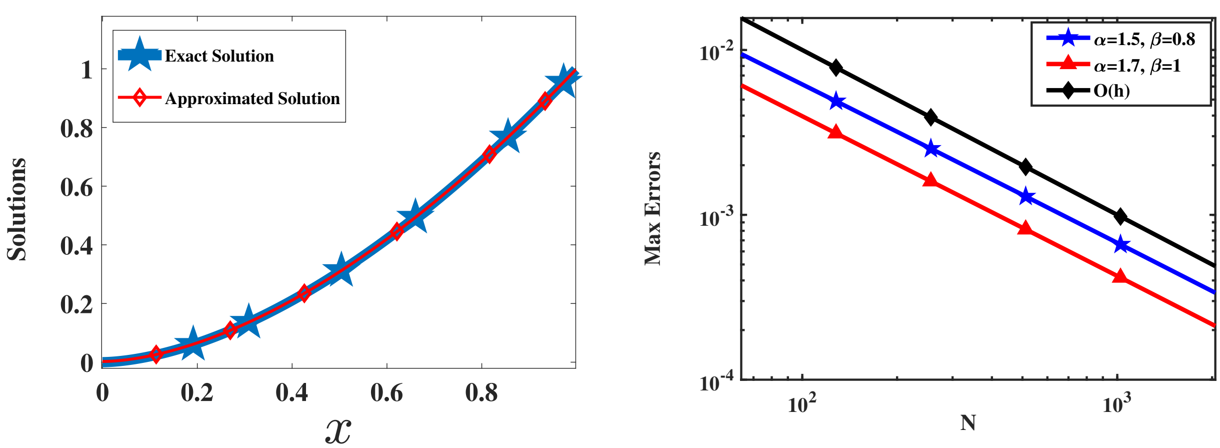
The numerical results for various values of α and β satisfying the condition $\alpha - \beta = 1$ are summarized in Tables 8–9 for Examples 6.1–6.3. These tables highlight the accuracy and behavior of the solution under different parameter settings. Additionally, the log-log plots in Figure 4 visually demonstrate the convergence rates and trends of the solution, providing further insight into the impact of parameters on the model.

7. Conclusion

In this paper, we have presented a scheme based on the L_1 method for solving a steady-state fractional convection-diffusion equation (FDE) with a reaction term. The model problem incorporates a mixed-fractional derivative as the higher-order derivative term. We have addressed the discrete maximum principle and conducted a thorough

TABLE 7. Maximum errors and orders of convergence obtained for $\beta = 1$ of Example 6.3.

$\beta = 1$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
$\alpha = 1.1$	7.0687e-03	3.4367e-03	1.6756e-03	8.1852e-04	4.0044e-04	1.9614e-04
C_N	1.0404	1.0364	1.0336	1.0314	1.0297	
$\alpha = 1.2$	6.2513e-03	3.0189e-03	1.4651e-03	7.1372e-04	3.4876e-04	1.7087e-04
C_N	1.0501	1.0430	1.0376	1.0331	1.0293	
$\alpha = 1.3$	5.8920e-03	2.8593e-03	1.3960e-03	6.8460e-04	3.3690e-04	1.6626e-04
C_N	1.0431	1.0344	1.0279	1.0229	1.0189	
$\alpha = 1.4$	5.8137e-03	2.8501e-03	1.4050e-03	6.9517e-04	3.4483e-04	1.7136e-04
C_N	1.0284	1.0204	1.0152	1.0115	1.0088	
$\alpha = 1.5$	5.8972e-03	2.9250e-03	1.4565e-03	7.2677e-04	3.6300e-04	1.8140e-04
C_N	1.0116	1.0059	1.0030	1.0015	1.0008	
$\alpha = 1.6$	6.0463e-03	3.0338e-03	1.5255e-03	7.6733e-04	3.8575e-04	1.9377e-04
C_N	0.9949	0.9918	0.9914	0.9922	0.9933	
$\alpha = 1.7$	6.1609e-03	3.1243e-03	1.5858e-03	8.0415e-04	4.0707e-04	2.0568e-04
C_N	0.9796	0.9783	0.9797	0.9822	0.9848	
$\alpha = 1.8$	6.1102e-03	3.1231e-03	1.5978e-03	8.1643e-04	4.1631e-04	2.1180e-04
C_N	0.9682	0.9669	0.9687	0.9717	0.9750	
$\alpha = 1.9$	5.7019e-03	2.9121e-03	1.4916e-03	7.6417e-04	3.9112e-04	1.9990e-04
C_N	0.9694	0.9652	0.9649	0.9663	0.9684	



(a) Solutions for $\alpha = 1.5$, $\beta = 0.8$, and $N = 256$.

(b) Log-log plots for various values of α and β.

FIGURE 3. Figures corresponding to Example 6.3.

error analysis. The convergence of the scheme was established using a carefully constructed discrete barrier function. Furthermore, semilinear FDEs were successfully solved by applying Newton's quasilinearization technique followed by the proposed scheme. Finally, several numerical examples were provided to validate the theoretical results.

ACKNOWLEDGEMENT

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TABLE 8. Errors and orders of convergence for Example 6.1 for various α and β with the condition $\alpha - \beta = 1$.

(α, β)	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
(1.7, 0.7)	6.2058e-03	3.1289e-03	1.5753e-03	7.9213e-04	3.9789e-04	1.9969e-04
C_N	0.9880	0.9900	0.9918	0.9934	0.9946	
(1.8, 0.8)	5.8864e-03	2.9733e-03	1.5002e-03	7.5615e-04	3.8072e-04	1.9152e-04
C_N	0.9853	0.9869	0.9885	0.9899	0.9913	
(1.9, 0.9)	5.4309e-03	2.7403e-03	1.3825e-03	6.9723e-04	3.5144e-04	1.7705e-04
C_N	0.9869	0.9870	0.9876	0.9883	0.9891	
(2, 1)	4.7587e-03	2.3743e-03	1.1859e-03	5.9263e-04	2.9624e-04	1.4810e-04
C_N	1.0031	1.0015	1.0008	1.0004	1.0002	

TABLE 9. Maximum differences and orders of convergence for Example 6.2 obtained by the double-mesh principle for various α and β with the condition $\alpha - \beta = 1$.

(α, β)	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
(1.7, 0.7)	1.6220e-03	8.3421e-04	4.2578e-04	2.1620e-04	1.0938e-04	5.5192e-05
$\widetilde{C}_N$	0.9593	0.9703	0.9778	0.9830	0.9868	
(1.8, 0.8)	1.3341e-03	6.8813e-04	3.5291e-04	1.8019e-04	9.1686e-05	4.6524e-05
$\widetilde{C}_N$	0.9551	0.9634	0.9698	0.9748	0.9787	
(1.9, 0.9)	1.0152e-03	5.0655e-04	2.5985e-04	1.3301e-04	6.7948e-05	3.4645e-05
$\widetilde{C}_N$	1.0031	0.9630	0.9661	0.9691	0.9718	
(2, 1)	1.2308e-03	6.2042e-04	3.1145e-04	1.5603e-04	7.8091e-05	3.9064e-05
$\widetilde{C}_N$	0.9883	0.9943	0.9972	0.9986	0.9993	

TABLE 10. Maximum errors and orders of convergence obtained for Example 6.3 for various α and β with the condition $\alpha - \beta = 1$.

(α, β)	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$	$N = 2^{10}$	$N = 2^{11}$
(1.7, 0.7)	8.8897e-03	4.5130e-03	2.2884e-03	1.1582e-03	5.8506e-04	2.9501e-04
C_N	0.9781	0.9797	0.9824	0.9853	0.9878	
(1.8, 0.8)	8.0870e-03	4.1308e-03	2.1084e-03	1.0742e-03	5.4607e-04	2.7702e-04
C_N	0.9692	0.9703	0.9730	0.9761	0.9791	
(1.9, 0.9)	6.7851e-03	3.4675e-03	1.7745e-03	9.0769e-04	4.6374e-04	2.3658e-04
C_N	0.9685	0.9665	0.9671	0.9689	0.9710	
(2, 1)	4.6414e-03	2.2945e-03	1.1408e-03	5.6876e-04	2.8398e-04	1.4189e-04
C_N	1.0164	1.0082	1.0041	1.0020	1.0010	

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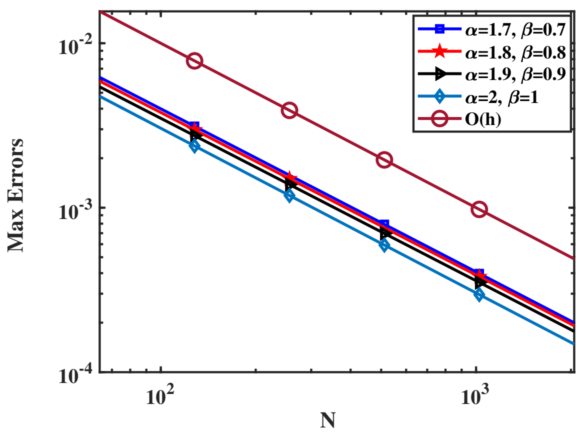
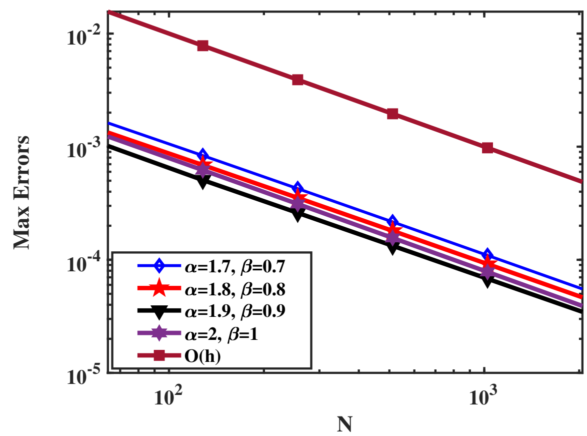
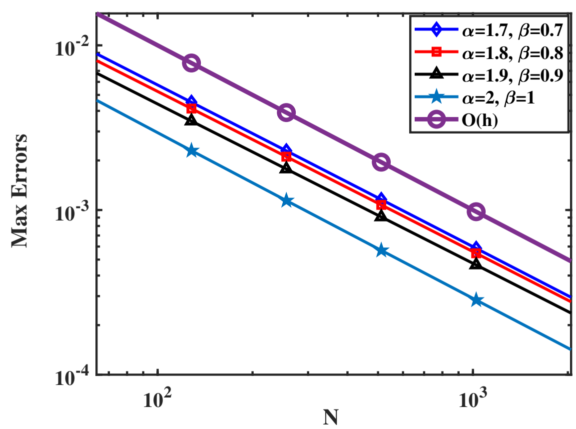


(a) Log-log plots for Example 6.1 with $\alpha - \beta = 1$.



(b) Log-log plots for Example 6.2 with $\alpha - \beta = 1$.



(c) Log-log plots for Example 6.3 with $\alpha - \beta = 1$.

FIGURE 4. Figures corresponding to Tables 8-10.

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