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Semi-analytical solutions for time-fractional Cauchy reaction-diffusion equations via new Elazki transform iterative method

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Abstract

In this article, the estimated analytic solutions for time-fractional Cauchy Reaction-Diffusion Equations (CRDE) are obtained using a New Elzaki Transform Iterative Method (NETIM). This method is the fusion of the Elzaki transform and the Iterative approach. The proposed technique is elegant and easy to adopt and comprehend. The semi-analytical results demonstrate, as this paper shows, a graphical interpretation of the solution using the mathematical software "Mathematica Wolform" and considering Caputo's sense derivatives to analytical results, the suggested strategy is efficient and straightforward.

Keywords. Caputo fractional derivative, Elzaki transform, Cauchy reaction-diffusion equations.
2010 Mathematics Subject Classification. 26A33, 35R11, 33E12.

1. Introduction

The fractional differential equations have fascinated mathematicians, physicists, and engineering researchers in recent decades [9, 12, 18] (Debnath 1997; Jiwari and Mittal 2011). Fractional derivatives and fractional calculus can be used to model various problems. But users need help finding the exact solutions they need. Most of the time, you have to use numerical methods and ways to get close to the answer. Fractional differential equations that are both linear and nonlinear have been solved in many ways so far. The time-fractional CRDE is one example.The Adomian decomposition method(ADM) (Wazwaz 1999) [28], homotopy analysis method(HAM), [16, 22], homotopy perturbation method(HPM) [11]. The fraction of time CRDE can be used to explain various types of systems, like linear and nonlinear in engineering, biology, ecology, chemistry, geo-hydrodynamics [4], and physics. (Britton 1998; Grindrop)[6]. Daftardar-gejji and Jafari devised the iterative skill in 2006 to solve linear and nonlinear fractional differential equations[7].

Many problems in fractional derivatives [21], hydrodynamics [1], chemical diffusion [30], option pricing [10], computational fluid dynamics [25], control theory [29], biological population model [19], quantitative analysis of sediments loss [24], generalized couette flow of couple stress nanofluid with heat, and mass transfer [2, 3] can be modelled using partial differential equations (PDEs). Finding numerical solutions to nonlinear PDEs has been focused on in recent years.

In this article, we solved the time-fractional heat-like and wave-like equations using a new Elzaki transform iterative method (NETIM) and NIM [13, 14, 17, 20, 27]. The benefit of this new method is that it makes the calculations easy and gives the closest possible answer [5, 8, 26].

In the present study, in operator form, the given time-fractional CRDE is thought of as [15]

$$D_{\omega}^{\beta} y(\xi,\omega) = v\frac{\partial^2 y(\xi,\omega)}{\partial \xi^2} + p(\xi,\omega)y(\xi,\omega), \xi \in R, \omega > 0, 0 < \beta \leq 1, \tag{1.1}$$

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Our goal in this paper is to approximate analytic solutions for time-fractional CRDE with appropriate initial conditions to apply (NETIM).

2. Background

Definition 2.1. [4] A Caputo fractional derivative of the function $y(a, b)$ is defined as,

$$D_a^{\beta} y(a,b) = \frac{1}{\Gamma(l-\beta)} \int_0^x (a-\jmath)^{(l-\beta-1)} y^{(l)}(\jmath, b) d\jmath, l-1 < \beta \leq l, l \in N. \tag{2.1}$$

$d^l \equiv \frac{d^l}{dx^l}$ and j_x^{β} - denote the R-L fractional integral operator of order $\beta > 0$ defined as $d^{\jmath} \equiv \frac{d^l}{dx^l}$ and j_x^{β} respectively.

$$J_a^{\beta} y(a,b) = \frac{1}{\Gamma\beta} \int_0^a (a-\jmath)^{(\beta-1)} y(\jmath, b) d\jmath, \jmath > 0, k-1 < \beta \leq k, k \in N. \tag{2.2}$$

Definition 2.2. [21] The mittag-leffler function is given by,

$$E_{\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\beta^k + 1)} \ (\beta \in c, re(\beta) > 0), \tag{2.3}$$

$E_{\beta,\alpha}$ is Mittag-Leffler function in two parameters.

$$E_{\beta,\alpha}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\beta^k + \alpha)} \ \beta, \alpha \in C, \ R(\beta), R(\alpha) > 0. \tag{2.4}$$

Definition 2.3. [31] The Elzaki transform of a function $g(p), p > 0$ is defined as

$$E[g(p)] = v \int_0^{\infty} e^{\frac{-p}{v}} g(p) dp, v \in (-T_1, T_2) \ and \ g(p) \in A, \tag{2.5}$$

where

$$A = \left\{ g(p)/\exists M, T_1, T_2 > 0, |g(p)| \leq M e^{\frac{|p|}{T_j}}, if \ p \in (-1)^j \times [0, \infty) \right\}. \tag{2.6}$$

Definition 2.4. [31] The Elzaki transform of the Caputo fractional derivative is defined as

$$E[D_{\xi}^{n\xi} y(\xi, \omega)] = v^{-n\beta} E[y(\xi, \omega)] - \sum_{k=0}^{n-1} v^{-n\beta+k+2} y^{(k)}(0, \omega), n-1 < n\beta < n. \tag{2.7}$$

3. The New Elzaki transform Iterative Method (NETIM)

To analyse this New Elzaki Iterative Transform Method [27, 32], we suppose a fractional partial differential equation with non-linear, non-homogenous and the initial conditions of the form:

$$D_{\omega}^{n\beta} y(\xi, \omega) + Ly(\xi, \omega) + R(y(\xi, \omega)) + = g(\xi, \omega), \qquad n-1 < n\beta \leq n, y(\xi, 0) = h(\xi). \tag{3.1}$$

where $D_{\omega}^{n\beta}$ is the Caputo operator fractional derivative, $D_{\omega}^{n\beta} = \frac{\partial^{n\beta}}{\partial \omega^{n\beta}}$, L and R are linear operator and nonlinear operator respectively, $g(\xi, \omega)$ is continuous function.

Operating the Elzaki transform on the Equation (3.1) we have

$$E\left[D_{\omega}^{n\beta} y(\xi, \omega)\right] + E\left[L(y(\xi, \omega)) + R(y(\xi, \omega))\right] = E\left[g(\xi, \omega\right], \tag{3.2}$$

using the property of Elzaki transformation we obtain

$$E\left[y(\xi, \omega)\right] - v^{n\beta} \sum_{k=0}^{n-1} y^k(\xi, 0) + v^{n\beta} E\left[Ly(\xi, \omega) + Ry(\xi, \omega)\right] - \left[g(\xi, \omega)\right] = 0. \tag{3.3}$$

Operate inverse Elzaki transform on the Eq. (3.3) we get

$$y(\xi,\omega) = E^{-1}[v^{n\beta}\sum_{k=0}^{n-1} y^k(\xi,0)] - E^{-1}[v^{n\beta}E[Ly(\xi,\omega) + Ry(\xi,\omega) - g(\xi,\omega)]]. \tag{3.4}$$

Next, assume that

$$f(\xi,\omega) = E^{-1}[v^{n\beta}\sum_{k=0}^{n-1} y^k(\xi,0) + v^{n\beta}E[g(\xi,\omega)]], \tag{3.5}$$

$$N(y(\xi,\omega)) = -E^{-1}[v^{n\beta}E[Ry(\xi,\omega)]], \tag{3.6}$$

$$K[y(\xi,\omega)] = -E^{-1}[v^{n\beta}E[Ly(\xi,\omega)]]. \tag{3.7}$$

Thus, Eq. (3.4) written like as

$$y(\xi,\omega) = f(\xi,\omega) + K(y(\xi,\omega)) + N(y(\xi,\omega)), \tag{3.8}$$

where f, K and N are a known function, linear and non-linear operators of, u respectively. The solution of equation is given in series form

$$y(\xi,\omega) = (\sum_{q=0}^{\infty} y(\xi,\omega)), \tag{3.9}$$

we have

$$K(\sum_{q=0}^{\infty} y(\xi,\omega)) = \sum_{q=0}^{\infty} K(y(\xi,\omega)).$$

The N is decomposed as, (N-non-linear operator)

$$N(\sum_{q=0}^{\infty} y_q) = N(y_0) + \left\{ N(\sum_{j=0}^{q} y_j) - N(\sum_{j=0}^{q-1} y_j) \right\}. \tag{3.10}$$

Therefore, Eq. (3.10)can be modified in recursive relation form , Defining the

$$\begin{aligned} y_0 =& f, \\ y_1 =& K(y_0) + N(y_0), \\ &\ldots\ldots \\ y_{q+1} =& K(y_q) + N(y_0 + \ldots + y_q). \end{aligned} \tag{3.11}$$

we have

$$(y_1 + y_2 + \ldots\ldots + y_{q+1}) = K(y_0 + \ldots. + y_q) + N(y_0 + \ldots. + y_q), \tag{3.12}$$

namely,

$$\sum_{q=0}^{\infty} y_q(\xi,\omega) = f + K(\sum_{q=0}^{\infty} y_q(\xi,\omega)) + N(\sum_{q=0}^{\infty} y_q(\xi,\omega)). \tag{3.13}$$

The q^{th}-term approximate is given by

$$y = y_1 + y_2 + \ldots\ldots + y_{q-1}. \tag{3.14}$$

4. Convergence and error analysis

Theorem 4.1. *Let $y_p(\xi,\omega)$ and $y_n(\xi,\omega)$ be the members of Banach space H, and the exact solution of Eq. (3.1) be $y(\xi,\omega)$. The Series solution $\sum_{p=0}^{\infty} y_p(\xi,\omega)$ converges to $y(\xi,\omega)$, if $y_p(\xi,\omega) \le \lambda y_{p-1}(\xi,\omega)$ for $\lambda \in (0,1)$, that is for any $y > 0, \exists E$ such that $||y_{p+n}(\xi,\omega)|| \le y, \forall p,n > E$.*

Proof. Let $u_p(\xi,\omega) = y_0(\xi,\omega)+y_1(\xi,\omega)+y_2(\xi,\omega)+\ldots+y_p(\xi,\omega)$ be the sequence of p^{th} partial sum of series $\sum_{p=0}^{\infty} y_p(\xi,\omega)$. Now consider

$$\begin{aligned} ||u_{p+1}(\xi,\omega) - u_p(\xi,\omega)|| &= ||y_{p+1}(\xi,\omega)|| \\ &\le \lambda||y_p(\xi,\omega)|| \\ &\le \lambda^2||y_{p-1}(\xi,\omega)|| \\ &\le \lambda^3||y_{p-2}(\xi,\omega)|| \\ &\vdots \\ &\le \lambda^{p+1}||y_0(\xi,\omega)||, \end{aligned} \tag{4.1}$$

for $\forall n, p \in E$.

Consider

$$\begin{aligned} ||u_p(\xi,\omega) - u_n(\xi,\omega)|| &= ||y_{p+n}(\xi,\omega)|| \\ &= ||\big(u_p(\xi,\omega) - u_{p-1}(\xi,\omega)\big) \\ &+ \big(u_{p-1}(\xi,\omega) - u_{p-2}(\xi,\omega)\big) + \big(u_{p-2}(\xi,\omega) - u_{p-3}(\xi,\omega)\big) + \ldots + \big(u_{n+1}(\xi,\omega) - u_n(\xi,\omega)\big)|| \\ &\le ||\big(u_p(\xi,\omega) - u_{p-1}(\xi,\omega)\big)|| + ||\big(u_{p-1}(\xi,\omega) - u_{p-2}(\xi,\omega)\big)|| + ||\big(u_{p-2}(\xi,\omega) - u_{p-3}(\xi,\omega)\big)|| \\ &+ \ldots + ||\big(u_{n+1}(\xi,\omega) - u_n(\xi,\omega)\big)|| \\ &\le \lambda^p||y_0(\xi,\omega)|| + \lambda^{p-1}||y_0(\xi,\omega)|| + \lambda^{p-2}||y_0(\xi,\omega)|| + \ldots + \lambda^{p-1}||y_0(\xi,\omega)|| \\ &= ||y_0(\xi,\omega)||\big(\lambda^p + \lambda^{p-1} + \ldots + \lambda^{p+1}\big) \\ &= ||y_0(\xi,\omega)||\big(\frac{1-\lambda^{p-n}}{1-\lambda}\big)\lambda^{n+1}. \end{aligned} \tag{4.2}$$

Since $0 < \lambda < 1$, and $y_0(\xi,\omega)$ is bounded, so assume that

$$y = ||y_0(\xi,\omega)||\big(\frac{1-\lambda^{p-n}}{1-\lambda}\big)\lambda^{n+1},$$

we get the desired result. Also, $\sum_{p=0}^{\infty} y_p(\xi,\omega)$ is a Cauchy sequence in H, which implies that there exists $y_0(\xi,\omega) \in H$ such that $\lim_{p\to\infty} y_p(\xi,\omega) = y(\xi,\omega)$. Hence prove. □

Theorem 4.2. *Let $\sum_{p=0}^{q} y_p(\xi,\omega)$ be the finite and approximate solution of $y(\xi,\omega)$. If $||y_{p+1}(\xi,\omega)|| \le \lambda||y_0(\xi,\omega)||$ for $\lambda \in (0,1)$, then the maximum absolute error is*

$$||y(\xi,\omega) - \sum_{p=0}^{q} y_p(\xi,\omega)|| \le \frac{\lambda^{q+1}}{1-\lambda}||y_0(\xi,\omega)||.$$

Proof.

$$\begin{aligned} ||y(\xi,\omega) - \sum_{p=0}^{q} y_p(\xi,\omega)|| &= ||\sum_{p=0}^{\infty} y_p(\xi,\omega)|| \\ &\leq \sum_{p=q+1}^{\infty} ||y_p(\xi,\omega)|| \\ &\leq \sum_{p=q+1}^{\infty} \lambda^q ||y_0(\xi,\omega)|| \lambda^{q+1}\big(1+\lambda+\lambda^2+\dots\big)||y_0(\xi,\omega)|| \\ &\leq \frac{\lambda^{q+1}}{1-\lambda}||y_0(\xi,\omega)||. \end{aligned} \tag{4.3}$$

□

5. Solutions of the time-fractional CRDE:

Example 5.1. We acknowlege the given linear time fractional (CRDE) [15]:

$$D_\omega^\beta y(\xi,\omega) = \frac{\partial^2 y(\xi,\omega)}{\partial \xi^2} - y(\xi,\omega), 0 < \beta \leq 1, \tag{5.1}$$

Subject to the initial condition

$$y(\xi,0) = e^{-\xi} + \xi. \tag{5.2}$$

Operating the Elzaki transform on the Eq. (5.1) and using the initial condition of Eq. (5.2) we get

$$E[y(\xi,\omega)] = \frac{e^{-\xi}+\xi}{v^2} + \frac{1}{v^{-\beta}} E\big[\frac{\partial^2 y}{\partial \xi^2} - y\big], \tag{5.3}$$

Operating the inverse Elzaki transform on the Eq. (5.3) we have

$$\begin{aligned} y(\xi,\omega) =& e^{-\xi} + \xi + E^{-1}\big[\frac{1}{v^{-\beta}} E[\frac{\partial^2 y}{\partial \xi^2} - y]\big], \\ & namely, \\ y(\xi,\omega) =& e^{-\xi} + \xi + E^{-1}\big[\frac{1}{v^{-\beta}} E[\frac{\partial^2 y}{\partial \xi^2} - y]\big]. \end{aligned} \tag{5.4}$$

According to the NETIM, we have

$$\begin{aligned} y_0 =& e^{-\xi} + \xi, \\ K[y(\xi,\omega)] =& E^{-1}\big[\frac{1}{v^{-\beta}} E[\frac{\partial^2 y}{\partial \xi^2} - y]\big]. \end{aligned} \tag{5.5}$$

By NETIM ,the given results are obtained

$$\begin{aligned} y_0(\xi,\omega) =& e^{-\xi} + \xi, \\ y_1(\xi,\omega) =& E^{-1}\big[\frac{1}{v^{-\beta}} E[\frac{\partial^2 y_0}{\partial \xi^2} - y_0]\big], \\ =& \xi \frac{(-\omega^\beta)}{\Gamma(\beta+1)}. \end{aligned} \tag{5.6}$$

$$\begin{aligned} y_2(\xi,\omega) =&E^{-1}\big[\frac{1}{v^{-\beta}}E[\frac{\partial^2(y_0+y_1)}{\partial\xi^2}]\big]-E^{-1}\big[\frac{1}{v^{-\beta}}y^2E[\frac{\partial^2 y_0}{\partial\xi^2}]\big] \\ =&\xi\big[\frac{(-\omega^\beta)^2}{\Gamma(2\beta+1)}+\frac{(-\omega^\beta)}{\Gamma(2\beta+1)}\big]-(\xi\frac{(-\omega^\beta)}{\Gamma(2\beta+1)}) \\ =&\xi\frac{(-\omega^\beta)^2}{\Gamma(2\beta+1)}. \end{aligned} \tag{5.7}$$

Therefore, the approximate analytical solution of the problem in the series form can be obtained as,

$$\begin{aligned} y(\xi,\omega) =&y_0(\xi,\omega)+y_1(\xi,\omega)+...,\\ y(\xi,\omega) =&e^{-\xi}+\xi+[\xi\frac{(-\omega^\beta)}{\Gamma(\beta+1)}+\xi\frac{(-\omega^\beta)^2}{\Gamma(2\beta+1)}+...] \\ =&e^{-\xi}+\xi E_\beta(-\omega^\beta), \end{aligned} \tag{5.8}$$

where $E_\beta(\omega^\beta)$ is mittage leffer function defined by Eq. (2.3).

Setting $\beta=1$, Eq. (5.1) becomes the following equation,

$$y(\xi,\omega)=\frac{\partial^2 y}{\partial\xi^2}-y, \tag{5.9}$$

with accurate solution

$$y(\xi,\omega)=e^{-\xi}+\xi e^{\omega}.$$

Remark 5.2. The time-fractional linear Cauchy equation for reaction and diffusion is shown above. Figures 1–6 offer the approximate solutions to the linear time fractional CRDE for $\beta=0.2$, 0.4, 0.6, 0.8, and 1 and the required solution for $\beta=1$. The answer is so easy to find that it depends on the values of time-fractional derivatives at all times.

Example 5.3. [23] We acknowledge the given linear time fractional CRDE

$$D_\omega^\beta y(\xi,\omega)=\frac{\partial^2 y(\xi,\omega)}{\partial\xi^2}-(1+4\xi^2)y(\xi,\omega), 0<\beta\le 1. \tag{5.10}$$

Subject to the initial condition

$$y(\xi,0)=e^{\xi^2}. \tag{5.11}$$

Operating the Elzaki transform on the Eq. (5.10) and using the initial condition of Eq. (5.11) we have,

$$E[y(\xi,\omega)]=\frac{e^{\xi^2}}{v^2}+\frac{1}{v^{-\beta}}E\big[\frac{\partial^2 y}{\partial\xi^2}-(1+4\xi^2)y\big]. \tag{5.12}$$

Operating the inverse Elzaki transform on the Eq. (5.12) we have

$$y(\xi,\omega)=e^{\xi^2}+E^{-1}\big[\frac{1}{v^{-\beta}}E\big[\frac{\partial^2 y}{\partial\xi^2}-(1+4\xi^2)y\big], \tag{5.13}$$

namely,

$$y(\xi,\omega)=e^{\xi^2}+E^{-1}\big[\frac{1}{v^{-\beta}}E\big[\frac{\partial^2 y}{\partial\xi^2}-(1+4\xi^2)y\big],$$

According to the NETIM, we have

$$\begin{aligned} y_0 =&e^{\xi^2},\\ K[y(\xi,\omega)] =&E^{-1}\big[\frac{1}{v^{-\beta}}E\big[\frac{\partial^2 y}{\partial\xi^2}-(1+4\xi^2)y\big]. \end{aligned} \tag{5.14}$$

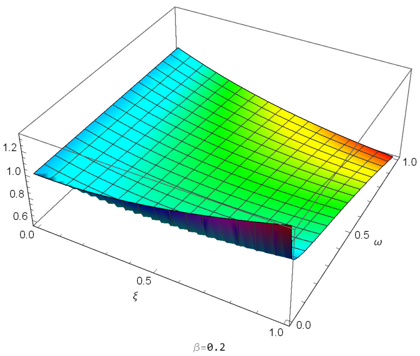

FIGURE 1. $\beta = 0.2$.

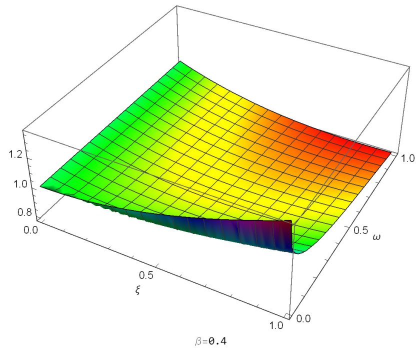

FIGURE 2. $\beta = 0.4$.

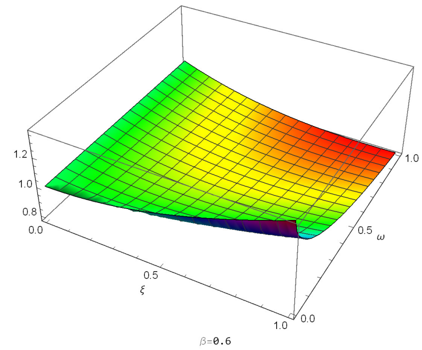

FIGURE 3. $\beta = 0.6$.

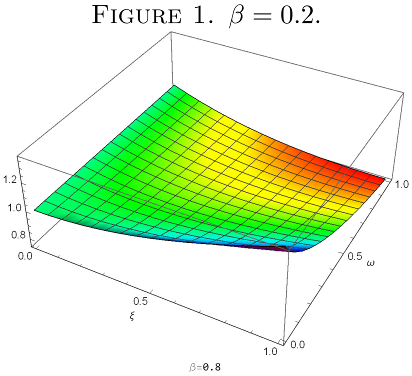

FIGURE 4. $\beta = 0.8$.

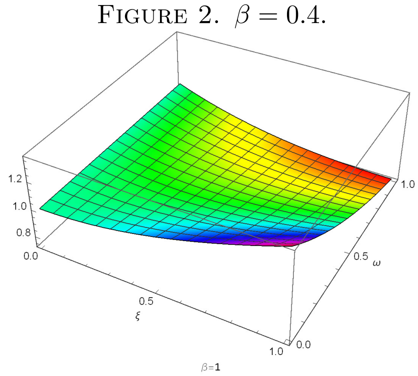

FIGURE 5. $\beta = 1.0$.

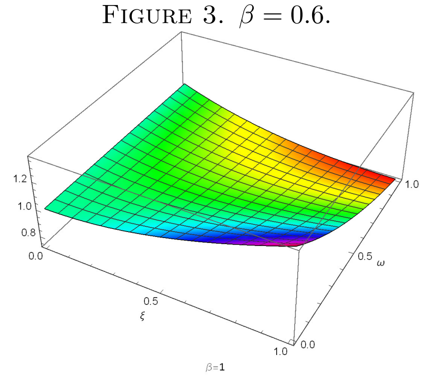

FIGURE 6. Exact.

By NETIM method ,the given result are obtained

$$
\begin{aligned}
&y_0(\xi,\omega) = y(\xi,0) = e^{\xi^2},\\
&y_1(\xi,\omega) = E^{-1}\Big[\frac{1}{\upsilon^{-\beta}}E\Big[\frac{\partial^2 y_0}{\partial \xi^2} - (1+4\xi^2)y_0\Big] \qquad (5.15)\\
&\qquad = e^{\xi^2}\frac{\omega^{\beta}}{\Gamma(\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
y_2(\xi,\omega) =& E^{-1}\Big[\frac{1}{\upsilon^{-\beta}}E\Big[\frac{\partial^2 (y_0+y_1)}{\partial \xi^2}\Big]\Big], -E^{-1}\Big[\frac{1}{\upsilon^{-\beta}}E\Big[\frac{\partial^2 (y_0)}{\partial \xi^2}\Big]\Big]\\
=& e^{\xi^2}\Big(\frac{\omega^{2\beta}}{\Gamma(2\beta+1)} - \frac{\omega^{\beta}}{\Gamma(\beta+1)}\Big) + e^{\xi^2}\frac{\omega^{\beta}}{\Gamma(\beta+1)} \qquad (5.16)\\
=& e^{\xi^2}\frac{\omega^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}
$$

Therefore, the approximate analytical solution of the problem in the series form can be obtained as,

$$
\begin{aligned}
y(\xi,\omega) =& y_0(\xi,\omega) + y_1(\xi,\omega) + ...,\\
y(\xi,\omega) =& e^{\xi^2}\big[1 + e^{\xi^2}\frac{(\omega^{\beta})}{\Gamma\beta+1} + e^{\xi^2}\frac{(\omega^{2\beta})}{\Gamma 2\beta+1} + ...\big] = e^{\xi^2}E_{\beta}(\omega^{\beta}), \qquad (5.17)
\end{aligned}
$$

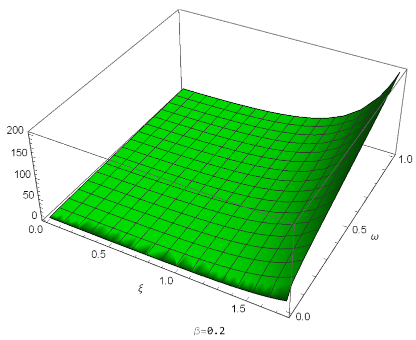

FIGURE 7. $\beta = 0.2$.

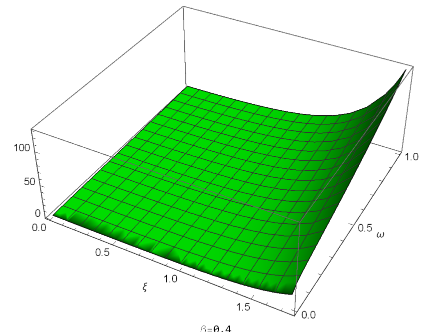

FIGURE 8. $\beta = 0.4$.

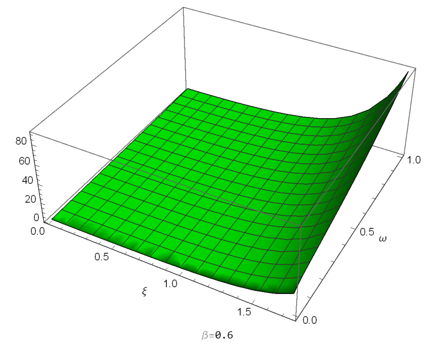

FIGURE 9. $\beta = 0.6$.

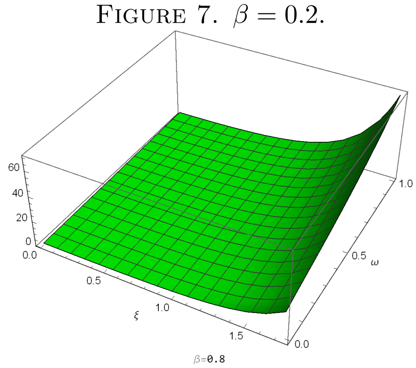

FIGURE 10. $\beta = 0.8$.

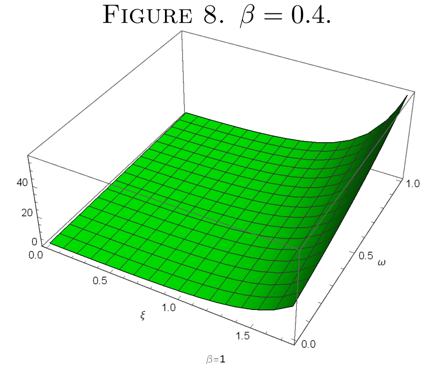

FIGURE 11. $\beta = 1.0$.

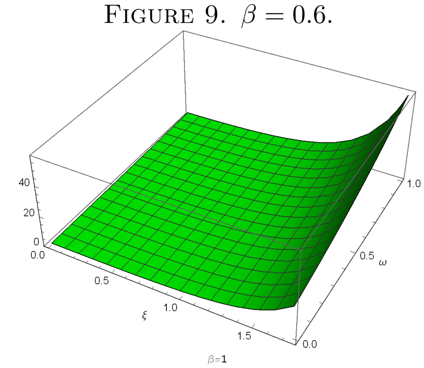

FIGURE 12. Exact.

where $E_\beta(\omega^\beta)$ is mittage leffer function defined by Eq. (2.3).

Setting $\beta = 1$, Eq. (5.10) become the equation,

$$y(\xi,\omega) = \frac{\partial^2 y}{\partial \xi^2} - (1 + 4\xi^2)y, \tag{5.18}$$

with accurate solution

$$y(\xi,\omega) = e^{\xi^2+\omega}.$$

Remark 5.4. The time-fractional linear Cauchy equation for reaction and diffusion is shown above. Figures 7–12 offer the approximate solutions of The linear time fractional CRDE shown above for different values of $\beta = 0.2, 0.4, 0.6, 0.8$, and 1 and the required solution for $\beta = 1$. The answer is easiest to find. It depends on the values of time-fractional derivatives at all times.

Example 5.5. [23] We acknowledge the given linear time fractional CRDE

$$D_\omega^\beta y(\xi,\omega) = \frac{\partial^2 y(\xi,\omega)}{\partial \xi^2} - (4\xi^2 - 2\omega + 2)y(\xi,\omega), 0 < \beta \le 1, \tag{5.19}$$

Subject to the initial condition

$$y(\xi, 0) = e^{\xi^2}. \tag{5.20}$$

Operating the Elzaki transform on the Eq. (5.19) and using the initial condition of Eq. (5.20) we have

$$E[y(\xi,\omega)] = \frac{e^{\xi^2}}{u^2} + \frac{1}{u^{-\beta}}E\Big[\frac{\partial^2 y}{\partial\xi^2} - (4\xi^2 - 2\omega + 2)y\Big]. \tag{5.21}$$

Operating the inverse Elzaki transform on the Eq. (5.20) we have,

$$y(\xi,\omega) = e^{\xi^2} + E^{-1}\Big[\frac{1}{v^{-\beta}}E\Big[\frac{\partial^2 y}{\partial\xi^2} - (4\xi^2 - 2\omega + 2)y\Big], \tag{5.22}$$

namely,

$$y(\xi,\omega) = e^{\xi^2} + E^{-1}\Big[\frac{1}{v^{-\beta}}E\Big[\frac{\partial^2 y}{\partial\xi^2} - (4\xi^2 - 2\omega + 2)y\Big],$$

According to the NETIM, we have

$$\begin{aligned} y_0 =& e^{\xi^2}, \\ K[y(\xi,\omega)] =& E^{-1}\Big[\frac{1}{v^{-\beta}}E\Big[\frac{\partial^2 y}{\partial\xi^2} - (4\xi^2 - 2\omega + 2)y\Big]. \end{aligned} \tag{5.23}$$

By NETIM method, the given result are obtained

$$\begin{aligned} y_0(\xi,\omega) =& y(\xi,0) = e^{\xi^2}, \\ y_1(\xi,\omega) =& E^{-1}\Big[\frac{1}{v^{-\beta}}E\Big[\frac{\partial^2 y_0}{\partial\xi^2} - (4\xi^2 - 2\omega + 2)y_0\Big] \\ =& 2e^{\xi^2}\frac{\omega^{\beta+1}}{\Gamma(\beta+2)}, \end{aligned} \tag{5.24}$$

$$\begin{aligned} y_2(\xi,\omega) =& E^{-1}\Big[\frac{1}{v^{-\beta}}E\Big[\frac{\partial^2 (y_0+y_1)}{\partial\xi^2}\Big]\Big] - E^{-1}\Big[\frac{1}{v^{-\beta}}E\Big[\frac{\partial^2 (y_0)}{\partial\xi^2}\Big]\Big] \\ =& 2e^{\xi^2}\Big(\frac{\omega^{\beta+1}}{\Gamma(\beta+2)} - \frac{\omega^{\beta}}{\Gamma(\beta+1)}\Big) + 2e^{\xi^2}\frac{\omega^{\beta}}{\Gamma(\beta+1)} \\ =& 4e^{\xi^2}\frac{(\beta+2)\omega^{2\beta+2}}{\Gamma(2\beta+3)}. \end{aligned} \tag{5.25}$$

Therefore, the approximate analytical solution of the problem in the series form can be obtained as,

$$\begin{aligned} y(\xi,\omega) =& y_0(\xi,\omega) + y_1(\xi,\omega) + ... \\ y(\xi,\omega) =& e^{\xi^2} + 2e^{\xi^2}\frac{(\omega^{\beta+1})}{\Gamma\beta+2} + 4e^{\xi^2}\frac{(\beta+2)(\omega^{2\beta+2})}{\Gamma 2\beta+3} + 8e^{\xi^2}\frac{(\beta+2)(2\beta+3)(\omega^{3\beta+3})}{\Gamma 3\beta+4} + ... \\ =& e^{\xi^2}E_\beta(\omega^{2\beta}), \end{aligned} \tag{5.26}$$

where $E_\beta(\omega^\beta)$ is Mittage Leffler function defined by Eq. (2.3).

Setting $\beta = 1$, Eq. (5.19) become the equation,

$$y(\xi,\omega) = \frac{\partial^2 y}{\partial\xi^2} - (4\xi^2 - 2\omega + 2)y, \tag{5.27}$$

with accurate solution

$$y(\xi,\omega) = e^{\xi^2+\omega^2}.$$

Remark 5.6. The time-fractional linear Cauchy equation for reaction and diffusion is shown above. Figures 13–18 offer the approximate solutions to The linear time CRDE for $\beta = 0.2$, 0.4, 0.6, 0.8, and 1 and the required solution for $\beta = 1$. The answer is so easiest to find that it depends on the values of time-fractional derivatives at all times.

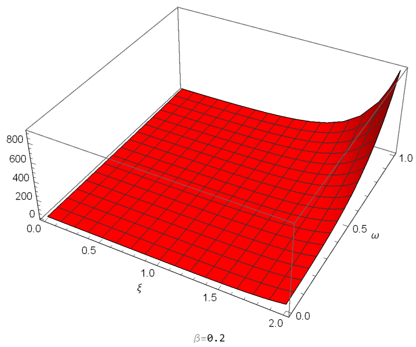

FIGURE 13. $\beta = 0.2$.

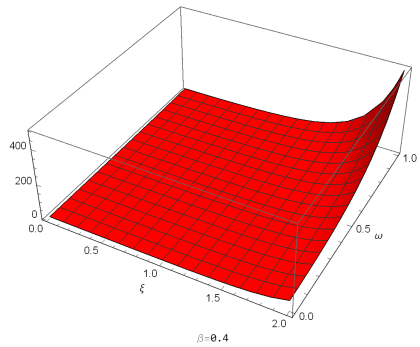

FIGURE 14. $\beta = 0.4$.

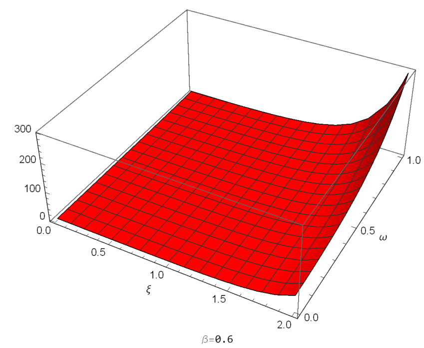

FIGURE 15. $\beta = 0.6$.

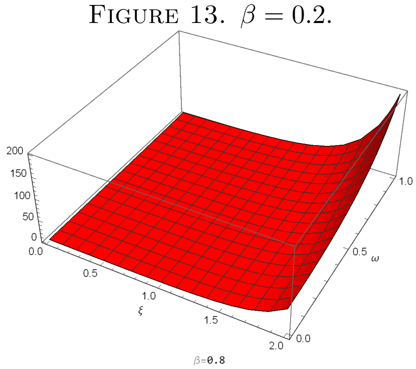

FIGURE 16. $\beta = 0.8$.

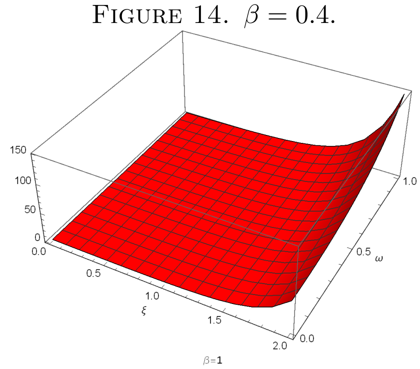

FIGURE 17. $\beta = 1.0$.

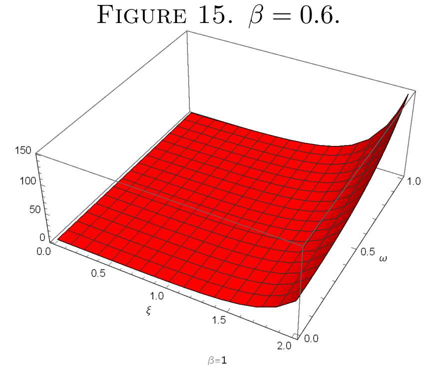

FIGURE 18. Exact.

Example 5.7. [23] We acknowledge the given linear time fractional CRDE

$$D_\omega^\beta y(\xi,\omega) = \frac{\partial^2 y(\xi,\omega)}{\partial \xi^2} + 2\omega y(\xi,\omega), 0 < \beta \leq 1. \tag{5.28}$$

Subject to the initial condition

$$y(\xi,0) = e^\xi. \tag{5.29}$$

Operating the Elzaki transform on the Eq. (5.28) and using the initial condition of Eq. (5.29) we get

$$E[y(\xi,\omega)] = \frac{e^\xi}{v^2} + \frac{1}{v^{-\beta}} E\big[\frac{\partial^2 y}{\partial \xi^2} + 2\omega y\big]. \tag{5.30}$$

Operating the inverse Elzaki transform on the Eq. (5.30) we have

$$y(\xi,\omega) = e^\xi + E^{-1}\big[\frac{1}{u^{-\beta}} E\big[\frac{\partial^2 y}{\partial \xi^2} + 2\omega y\big]\big], \tag{5.31}$$

namely,

$$y(\xi,\omega) = e^\xi + E^{-1}\big[\frac{1}{v^{-\beta}} E\big[\frac{\partial^2 y}{\partial \xi^2} + 2\omega y\big]\big].$$

TABLE 1. Compare the solution with the 10^{th} order approximation calculation of Eq. (5.1) and the accurate calculation for β=1.

	β						
(ξ, ω)	0.2	0.4	0.6	0.8	$Y_{NETIM}(1)$	$Y_{Exact}(1)$	$\|Y_{Exact} - Y_{NETIM}\|$
(0.2,0.3)	0.919727	0.932625	0.942174	0.953907	0.966894	0.966894	3.16802×10^{-13}
(0.4,0.5)	0.843819	0.875931	0.883486	0.895248	0.912932	0.940037	5.14745×10^{-11}
(0.6,0.7)	0.75944	0.832719	0.833698	0.83597	0.846763	0.918048	1.46326×10^{-8}
(0.8,0.9)	0.651812	0.797536	0.794016	0.78052	0.774585	0.900045	1.77552×10^{-8}

By NETIM method ,the given result are obtained

$$\begin{aligned} &y_0 = e^{\xi}, \\ &K[y(\xi,\omega)] = E^{-1}\big[\frac{1}{v^{-\beta}}E\big[\frac{\partial^2 y}{\partial \xi^2} + 2\omega y\big]. \end{aligned} \tag{5.32}$$

By iterative method ,the following result are obtained

$$\begin{aligned} y_0(\xi,\omega) =& y(\xi,0) = e^{\xi}, \\ y_1(\xi,\omega) =& E^{-1}\big[\frac{1}{v^{-\beta}}E\big[\frac{\partial^2 y_0}{\partial \xi^2} + 2\omega y_0\big]\big] \\ =& e^{\xi}\big(\frac{\omega^{\beta}}{\Gamma(\beta+1)} + \frac{2\omega^{\beta+1}}{\Gamma(\beta+2)}\big), \end{aligned} \tag{5.33}$$

$$\begin{aligned} y_2(\xi,\omega) =& E^{-1}\big[\frac{1}{v^{-\beta}}E[\frac{\partial^2 (y_0+y_1)}{\partial \xi^2}]\big] - E^{-1}\big[\frac{1}{u^{-\beta}}E[\frac{\partial^2 (y_0)}{\partial \xi^2}]\big] \\ =& e^{\xi}\big(\frac{\omega^{2\beta}}{\Gamma(2\beta+1)} + \frac{2(\beta+2)\omega^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{4(\beta+2)\omega^{2\beta+2}}{\Gamma(2\beta+3)}\big). \end{aligned} \tag{5.34}$$

Therefore, the approximate analytical solution of the problem in the series form can be obtained as,

$$\begin{aligned} y(\xi,\omega) =& y_0(\xi,\omega) + y_1(\xi,\omega) + ..., \\ y(\xi,\omega) =& e^{\xi} + e^{\xi}\big(\frac{\omega^{\beta}}{\Gamma(\beta+1)} + \frac{2\omega^{\beta+1}}{\Gamma(\beta+2)}\big) + e^{\xi}\big(\frac{\omega^{2\beta}}{\Gamma(2\beta+1)} + \frac{2(\beta+2)\omega^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{4(\beta+2)\omega^{2\beta+2}}{\Gamma(2\beta+3)}\big) + ... \\ =& e^{\xi}E_{\beta}(\omega^{\beta} + \omega^{2\beta}), \end{aligned} \tag{5.35}$$

where $E_{\beta}(\omega^{\beta})$ is Mittage Leffler function defined by Eq. (2.3). Setting $\beta = 1$, Eq. (5.28) becomes the equation

$$y(\xi,\omega) = \frac{\partial^2 y}{\partial \xi^2} + 2\omega y, \tag{5.36}$$

with accurate solution

$$y(\xi,\omega) = e^{\xi+\omega+\omega^2}. \tag{5.37}$$

Remark 5.8. The time-fractional linear Cauchy equation for reaction and diffusion is shown above. Figures 19–24 offer the approximate solutions to The linear time CRDE for $\beta = 0.2$, 0.4, 0.6, 0.8, and 1 and the required solution for $\beta = 1$. The answer is so easiest to find that it depends on the values of time-fractional derivatives at all times.

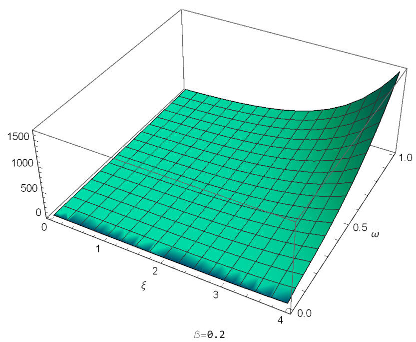

FIGURE 19. $\beta = 0.2$.

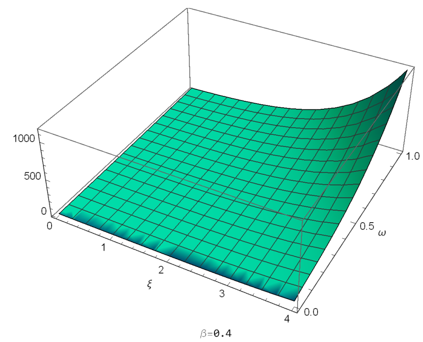

FIGURE 20. $\beta = 0.4$.

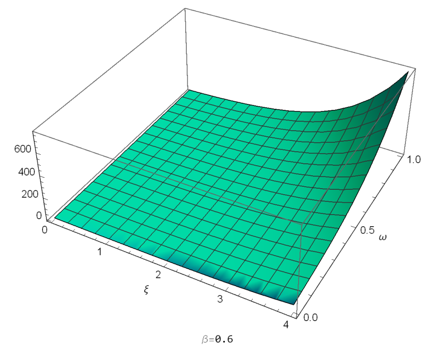

FIGURE 21. $\beta = 0.6$.

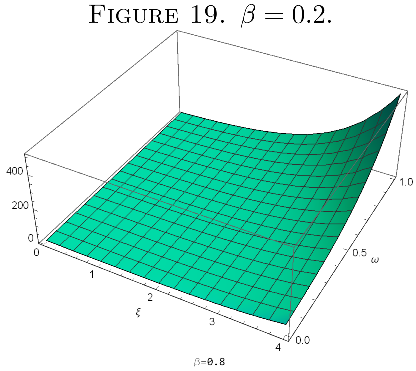

FIGURE 22. $\beta = 0.8$.

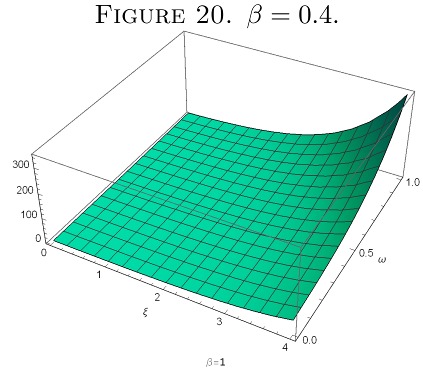

FIGURE 23. $\beta = 1.0$.

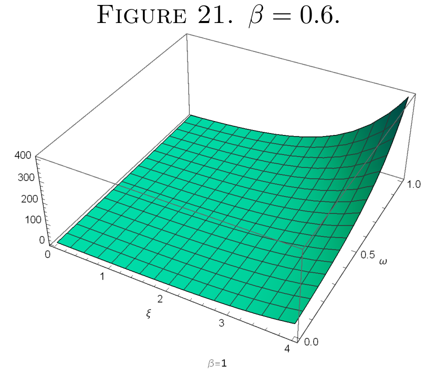

FIGURE 24. Exact.

TABLE 2. Compare the solution with the 10^{th} order approximation calculation of Eq. (5.10)and the accurate calculation for β=1.

	β						
(ξ,ω)	0.2	0.4	0.6	0.8	$Y_{NETIM}(1)$	$Y_{Exact}(1)$	$\lvert Y_{Exact} - Y_{NETIM}\rvert$
(0.2,0.3)	4.51611	2.59859	1.92565	1.59604	1.40495	1.40495	1.741015×10^{-12}
(0.4,0.5)	6.75882	3.91477	2.81346	2.26262	1.93479	1.71601	2.93381×10^{-10}
(0.6,0.7)	10.3077	6.16201	4.35585	3.43815	2.88637	2.09594	8.6493×10^{-9}
(0.8,0.9)	16.4298	10.2893	7.21822	5.62638	4.66459	2.55998	1.08849×10^{-7}

TABLE 3. Compare the solution with the 10^{th} order approximation calculation of Eq. (5.19)and the accurate calculation for β=1.

	β						
(ξ,ω)	0.2	0.4	0.6	0.8	$Y_{NETIM}(1)$	$Y_{Exact}(1)$	$\lvert Y_{Exact} - Y_{NETIM}\rvert$
(0.2,0.3)	1.73885	1.44206	1.28613	1.19511	1.13883	1.13883	5.19939×10^{-8}
(0.4,0.5)	3.46067	2.43033	1.93841	1.66983	1.50681	1.33643	9.96264×10^{-6}
(0.6,0.7)	8.36514	5.05954	3.54042	2.77389	2.33928	1.69893	3.7×10^{-4}
(0.8,0.9)	22.435	12.4614	7.75524	5.46862	4.25676	2.33965	6.4×10^{-3}

Table 4. Compare the solution with the 10^{th} order approximation calculation of Eq. (5.28) and the accurate calculation for β=1.

	β						
(ξ, ω)	0.2	0.4	0.6	0.8	$Y_{NETIM}(1)$	$Y_{Exact}(1)$	$\lvert Y_{Exact} - Y_{NETIM} \rvert$
(0.2,0.3)	6.59189	3.98393	2.69147	2.05623	1.72576	1.80399	0.078224
(0.4,0.5)	14.1542	8.70024	5.56162	3.86536	2.94868	2.58571	0.209508
(0.6,0.7)	28.5516	18.3435	11.7039	7.7702	5.5245	4.01485	0.464953
(0.8,0.9)	54.6337	37.0208	24.2375	15.9735	10.9595	6.75309	1.34544

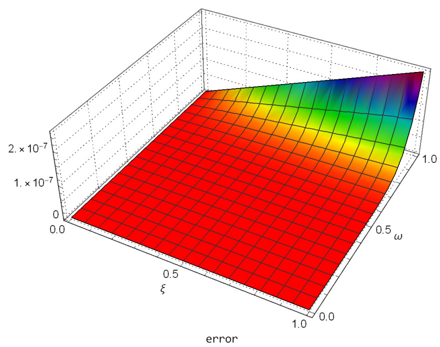

Figure 25. $\beta = 0.2$

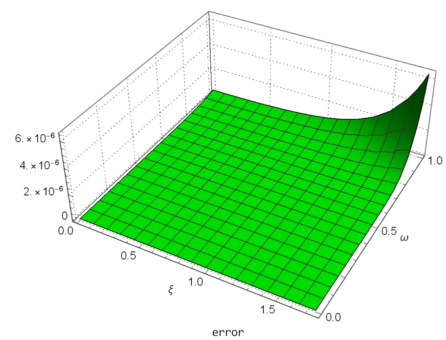

Figure 26. $\beta = 0.4$.

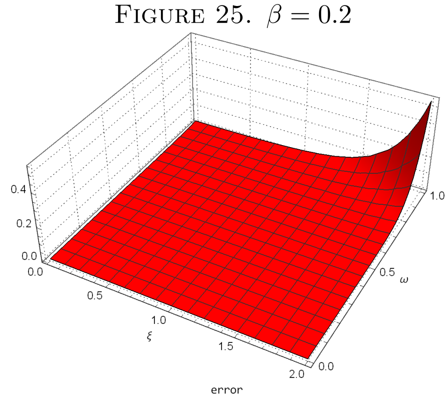

Figure 27. $\beta = 0.6$.

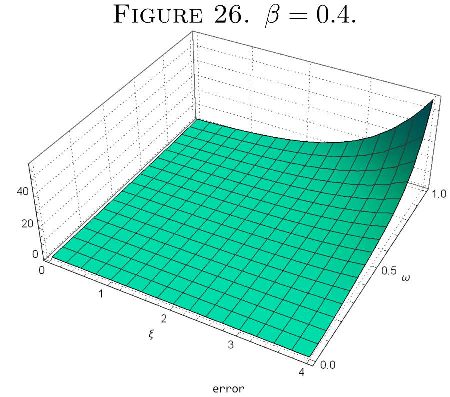

Figure 28. $\beta = 0.8$.

Remark 5.9. Figures 25–28 depict the absolute error between approximate and accurate calculation for β=1; by comparison, it is clear that by computing additional iterations, the efficiency and accuracy of this method (NETIM)can be significantly improved. We use a few iterations in this post. The precision of the estimated solution will be substantially enhanced if we employ additional iterations. As a result, the recommended method for solving the linear differential equation is precise and efficient.

6. Conclusion

This study found an approximation for the linear time fractional Cauchy Reaction-Diffusion Equation (CRDE) using the new Elzaki transform iterative method (NETIM). NETIM combines the new iterative method (NIM) and the Elzaki transform to get accurate and close analytical solutions for the time-fractional linear CRDE. The numbers show that the NETIM is faster, more accurate, and requires less calculation than other methods.

The method will help scientists and researchers who work on the subject of partial and ordinary differential equations a lot because it can cut down on the amount of work that needs to be done when compared to traditional methods. It also gives very accurate numerical results. The main benefit of the process is how quickly it gets to the answer. The numbers found here are in line with its higher level of exactness. It can be said that the NETIM method is so effective and powerful at finding semi-analytical solutions.

Conflict of interests

The authors declare that there is no conflict of interests.

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References

[1] S. Abbasbandy and A. Shirzadi, *An unconditionally stable finite difference scheme for equations of conservation law form,* Italian Journal of Pure and Applied Mathematics, *37* (2017), 1–4.
[2] F. Ali, A. Zubair, M. Arif, I. Khan, and K. S. Nisar, *A time fractional model of generalized Couette flow of couple stress nanofluid with heat and mass transfer: applications in engine oil,* IEEE Access, *8* (2020), 146944–146966.
[3] M. Arif, F. Ali, I. Khan, and K. Nisar, *A time fractional model with non–singular kernel the generalized Couette flow of couple stress nanofluid,* IEEE Access, *8* (2020), 77378–77395.
[4] A. Atangana, *Fractional operators with constant and variable order with application to geo–hydrology,* Academic Press, (2017).
[5] S. Bhalekar and V. Daftardar-Gejji, *Solving fractional–order logistic equation using a new iterative method,* International Journal Of Differential Equations, *2012* (2012), 12.
[6] N. Britton, *Reaction-diffusion equations and their applications to biology,* (Academic Press/Harcourt Brace Jovanovich Publishers, 1998.
[7] V. Daftardar-Gejji and H. Jafari, *An iterative method for solving nonlinear functional equations,* Journal of mathematical analysis and applications, *316*(2) (2006), 753–763.
[8] S. Das, *Analytical solution of a fractional diffusion equation by variational iteration method,* Computers And Mathematics With Applications, *57*(3) (2009), 483–487.
[9] L. Debnath, *Nonlinear partial differential equations for scientist and engineers,* Birkhauser, 1997.
[10] M. Dehghan and S. Pourghanbar, *Solution of the Black-Scholes equation for pricing of barrier option,* Zeitschrift Für Naturforschung A, *66* (2011), 289–296.
[11] J. He, Homotopy perturbation method: a new nonlinear analytical technique, Applied Mathematics And Computation, *135* (2003), 73–79.
[12] R. Jiwari, *A higher order numerical scheme for singularly perturbed Burger–Huxley equation*, J. Appl. Math. Inform., *29* (2011), 813–829.
[13] H. Khan, A. Khan, P. Kumam, D. Baleanu, and M. Arif, *An approximate analytical solution of the Navier–Stokes equations within Caputo operator and Elzaki transform decomposition method,* Advances In Difference Equations, *2020* (2020), 1–23.
[14] K. Kshirsagar, V. Nikam, S. Gaikwad, and S. Tarate, *The double fuzzy Elzaki transform for solving fuzzy partial differential equations,* Journal Of The Chungcheong Mathematical Society. *35* (2022), 177–196.
[15] S. Kumar, *A new fractional modeling arising in engineering sciences and its analytical approximate solution,* Alexandria Engineering Journal. *52* (2013), 813–819.
[16] S. Liao, *On the homotopy analysis method for nonlinear problems*, Appl Math Comput, *147* (2004), 499–513.
[17] J. Manafian and M. Lakestani, *Lump-type solutions and interaction phenomenon to the bidirectional Sawada–Kotera equation,* Pramana, *92* (2019), 1–13.

[18] R. C. Mittal and R. Jiwari, *Numerical study of two-dimensional reaction–diffusion Brusselator system by differential quadrature method,* International Journal for Computational Methods in Engineering Science and Mechanics, *12*(1) (2011), 14–25.

[19] K. Nisar, A. Ciancio, K. Ali, M. Osman, C. Cattani, D. Baleanu, A. Zafar, M. Raheel, and M. Azeem, *On beta–time fractional biological population model with abundant solitary wave structures,* Alexandria Engineering Journal, *61* (2022), 1996–2008.

[20] R. Nuruddeen, *Elzaki decomposition method and its applications in solving linear and nonlinear Schrodinger equations,* Sohag Journal Of Mathematics, *4* (2017), 1–5.

[21] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications,* Elsevier Science, (1998).

[22] M. Safari, D. Ganji, and M. Moslemi, *Application of He's variational iteration method and Adomian's decomposition method to the fractional KdV–Burgers–Kuramoto equation,* Comput Appl, *58* (2009), 2091–2097.

[23] S. Sharma and R. Bairwa, *Iterative Laplace Transform Method for Solving Fractional Heat and Wave–Like Equations,* Research Journal Of Mathematical And Statistical Sciences, *3* (2015), 4–9.

[24] N. Sheikh, M. Jamil, D. Ching, I. Khan, M. Usman, and K. Nisar, *A generalized model for quantitative analysis of sediments loss: A Caputo time fractional model,* Journal Of King Saud University–Science, *33* (2021), 101179.

[25] N. Smaoui and M. Zribi, *Dynamics and control of the seven–mode truncation system of the 2–d Navier–Stokes equations,* Communications In Nonlinear Science And Numerical Simulation, *32* (2016), 169–189.

[26] S. Tarate, A. Bhadane, S. Gaikwad, and K. Kshirsagar, *Solution of time–fractional equations via Sumudu–Adomian decomposition method,* Computational Methods For Differential Equations, *11* (2023), 345–356.

[27] S. Tarate, A. Bhadane, S. Gaikwad, and K. Kshirsagar, *Sumudu–iteration transform method for fractional telegraph equations,* J. Math. Comput. Sci., *12* (2022), Article ID 127.

[28] A. Wazwaz, *A reliable modification of Adomian decomposition method,* Appl Math Comput., *102* (1999), 77–86.

[29] B. Yildiz, O. Kilicoglu, and G. Yagubov, *Optimal control problem for nonstationary Schrödinger equation,* Numerical Methods For Partial Differential Equations: An International Journal, *25* (2009), 1195–1203.

[30] X. Zhong, J Vrijmoed, E. Moulas, and L. Tajčmanová, *A coupled model for intragranular deformation and chemical diffusion,* Earth And Planetary Science Letters, *474* (2017), 387–396.

[31] D. Ziane and M. H. Cherif. *Resolution of nonlinear partial differential equations by Elzaki transform decomposition method,* J. Appro. Theo. Appl. Math, *5* (2015), 17–30.

[32] D. Ziane, *Elzaki transform and the decomposition method for nonlinear fractional partial differential equations,* Int. J. Open Problems Compt. Math, *9*(4) (2016).