

## A higher-order orthogonal collocation technique for discontinuous two-dimensional problems with Neumann boundary conditions

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### Abstract

In this paper, the orthogonal spline collocation method (OSCM) is employed to address the solution of the Helmholtz equation in two-dimensional problems. It is characterized by discontinuous coefficients with certain wave numbers. The solution is approximated by employing distinct basis functions, namely, monomial along the x-direction and Hermite along the y-direction. Additionally, to solve the two-dimensional problems efficiently in the sense of computational cost with fewer operation counts, the matrix decomposition algorithm (MDA) is used to convert them into a set of one-dimensional problems. As a consequence, the resulting reduced matrix becomes non-singular in discrete cases. To assess the performance of the proposed numerical scheme, a grid refinement analysis is conducted to incorporate various wave coefficients of the Helmholtz equation. The illustrations and examples demonstrate a higher order of convergence compared to existing methods.

**Keywords.** Helmholtz equation, Orthogonal spline collocation methods, Matrix decomposition algorithm.

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### 1. INTRODUCTION

Helmholtz interface problems are encountered in numerous scientific and physical phenomena, such as the study of sound waves traveling from one medium to another [26, 29], radiation from the source of an electromagnetic field [30], fluid-solid interaction [12] and many more. There are various methods available in the literature to deal with Helmholtz equation, namely, the finite difference method [21] and the finite element method [24]. In computing the solution, these methods require a sufficiently small step size which results in a large number of algebraic equations and expensive computational costs. These equations form sparse matrices that require  $n^3$  operations to reach the solution. To refrain from indulging in numerous computations involving large matrices, the orthogonal spline collocation (OSC) method [9] can be used as a powerful and efficient method that reduces the number of operations to  $n$ . The well-established scheme exhibits several advantages when compared to existing iterative methods such as finite difference and boundary element methods. Some of these advantages include the following.

- The scheme easily manages the discontinuities that emerge in interface problems by using monomial basis functions [32].
- It can be applied to various types of boundary conditions on nonuniform meshes without sacrificing precision.
- The incorporation of high-order polynomial approximation results in a superior order of convergence, contributing to increased accuracy in the solution.

In the past decade, scientists and researchers have preferred OSC method due to its significant applicability with respect to its ease of implementation. De Boor and Swartz [13] first proposed the OSC method for linear ODEs with boundary value problems, and this inspired Fairweather and Meade [17] to further develop the solver to generate

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a sequence of approximate solutions that made a large impact on solving PDEs with different types of boundary conditions. In connection to this, the following authors in the articles [6–8, 15, 27, 28] implemented OSC method for elliptic and parabolic PDEs with mixed type boundary conditions. More specifically, as this method generates high optimal accuracy with super-convergence results, it is worth mentioning that the authors of these articles [2–4] have significantly worked on OSC method to solve Helmholtz equation over compact finite difference schemes [18, 20, 21]. One can refer to the article [5], where the authors have utilized the OSC method to derive approximate solutions for nonlinear boundary valued problems. The practical applicability of OSC method is that the discontinuity that occurs at any interior point of the domain can be tackled by considering monomial cubic basis for each divided sub-interval of the domain. In this regard, some of the applications of OSC method can also be seen in the recent articles [1, 19, 33]. Due to its stability for high order computation, the orthogonal spline collocation method effectively calculates the estimated solution of the Helmholtz equation. Motivated by the developments in OSCM, we have considered the two-dimensional Helmholtz equation,

$$u_{xx} + u_{yy} + \gamma_0^2 p(x)u = f(x, y), \quad (x, y) \in \Omega, \quad (1.1)$$

$$u(0, y) = 0, \quad u(1, y) = 0, \quad \text{on } \partial\Omega, \quad (1.2)$$

$$u_y(x, 0) = 0, \quad u_y(x, 1) = 0, \quad \text{on } \partial\Omega, \quad (1.3)$$

where  $\Omega = (a, b) \times (c, d)$  is a rectangular domain,  $\gamma_0^2$  is a wave number or material coefficient, and  $p(x)$  is a piecewise continuous function having finite jumps across a straight line  $\Gamma_i = \{(x, y) : x = x_i\}$  in  $\Omega$ .

The present focus of this article is to apply the orthogonal spline collocation method to the Helmholtz equation with Neumann boundary condition that extends the work of Feng [18]. Neumann boundary conditions, which signify the derivative of the solution at the boundary points, have several challenges while applying conventional numerical schemes like finite difference and finite element approaches to deal with the Helmholtz equation. The OSC method effectively handles these boundary conditions by incorporating them directly into the collocation process, leading to better accuracy and stability. Consideration of higher degree polynomials generates high order convergence of solutions. By choosing suitable basis functions, the OSC method can significantly reduce the computational cost associated with solving sparse linear systems of equations, which arises from the Helmholtz equation with Neumann boundary conditions. In particular, the MDA algorithm [10] and the almost block diagonal (ABD) solver [14] are very useful processes in obtaining the solution set.

For the notational convenience, we have assumed that  $w^2 = \gamma_0^2 p(x)$ . Further, the solution  $u(x, y)$  is assumed to satisfy the following natural jump conditions across the interface  $\Gamma_i$ :

$$[u] = \alpha, \quad [u_x] = \beta, \quad [u_y] = 0. \quad (1.4)$$

This might occur for the second or higher order partial derivatives of  $u(x, y)$  and the source function  $f(x, y)$  with respect to  $x$ . To this end, the article is organized as follows: Section 2 describes basic notations and the formulation of OSCM to solve the Equations (1.1)–(1.3). In section 3, the matrix decomposition algorithm is applied to the linear system of collection equations, which gives an almost block diagonal matrix. Finally, the performance of this method for Helmholtz equation for different values of wave number with respect to the Neumann boundary conditions is carried out in the numerical section 4.

## 2. ORTHOGONAL SPLINE COLLOCATION METHOD

In this section, OSCM with monomial cubic basis is applied in the  $x$ -direction and Hermite cubic basis functions in the  $y$ -direction to obtain the collocation solution of Eqs. (1.1)–(1.3). In this procedure, the following assumptions have been considered.

Suppose  $\pi_x = \{x_i\}_{i=0}^N$  and  $\pi_y = \{y_j\}_{j=0}^N$  denote the partitions of  $[a, b]$  and  $[c, d]$  such that  $x_i = ih_x$ ,  $i = 0, \dots, N$ , and  $y_j = jh_y$ ,  $j = 0, \dots, N$ , where  $h_x$  and  $h_y$  are the step sizes. Then we set  $I_i = [x_{i-1}, x_i]$  and  $I_j = [y_{j-1}, y_j]$ , where  $i, j = 1, \dots, N$ . Further, we define

$$\mathcal{M}_{-1}^3(\pi_x) = \{\Phi : \Phi \in PC[0, 1], \Phi|_{I_i} \in P_3, i = 1, 2, \dots, N\},$$



where  $PC[0, 1]$  represents the space of continuous piecewise functions on  $[0, 1]$  and  $P_3$  is a polynomial of degree  $\leq 3$ . Similarly,

$$\begin{aligned}\mathcal{M}_1^3(\pi_y) &= \{\chi : \chi \in C^1[0, 1], \chi|_{I_j} \in P_3, j = 1, 2, \dots, N\}, \\ \mathcal{M}_{-1}^{3,0}(\pi_x) &= \mathcal{M}_{-1}^3(\pi_x) \cap \{\chi : \chi(0) = \chi(1) = 0\}, \\ \mathcal{M}_1^{3,0}(\pi_y) &= \mathcal{M}_1^3(\pi_y) \cap \{\chi : \chi(0) = \chi(1) = 0\}.\end{aligned}$$

Next, the collocation points considered as  $\{\xi_i\}_{i=1}^{2N}$  on  $[0, 1]$  are actually two-point Gauss-Legendre quadrature points, which are defined by

$$\xi_{2i-1} = x_{i-1} + \frac{1}{2} \left(1 - \frac{1}{\sqrt{3}}\right) h_i, \quad \text{and} \quad \xi_{2i} = x_{i-1} + \frac{1}{2} \left(1 + \frac{1}{\sqrt{3}}\right) h_i, \quad i = 1, 2, \dots, N.$$

If  $x = \xi_{2i-1}$ , then  $x - x_{i-1} = \frac{1}{2}(1 + \rho_1)h$ , where  $\rho_1 = -\frac{1}{\sqrt{3}}$ . Likewise, if  $x = \xi_{2i}$ , then we have  $x - x_{i-1} = \frac{1}{2}(1 + \rho_2)h$ , where  $\rho_2 = -\rho_1$ .

To arrive at an approximate solution for the problems in Eqs. (1.1) and (1.3) through OSCM estimation, we define  $U \in \mathcal{M}_{-1}^{3,0} \otimes \mathcal{M}_1^{3,0}$  such that

$$U_{xx}(\xi_i, \xi_j) + U_{yy}(\xi_i, \xi_j) + w^2 U(\xi_i, \xi_j) = f(\xi_i, \xi_j), \quad i, j = 1, 2, 3, \dots, 2N. \quad (2.1)$$

Let the monomial basis functions for  $\mathcal{M}_{-1}^{3,0}$  be as follows:

$$\begin{aligned}\{\Phi_m\}_{m=1}^{4N+2} &= \{\Phi_1, \Phi_2, \dots, \Phi_m, \Phi_{m+1}, \dots, \Phi_{4N}, \Phi_{4N+1}, \Phi_{4N+2}\}, \\ \{\Phi_{4(k-1)+j}\}_{k=1}^N &= (x - x_{k-1})^{j-1}, \quad j = 1, 2, 3, 4 \quad \text{and} \quad \Phi_{4N+1} = 1, \quad \Phi_{4N+2} = (x - x_N),\end{aligned}$$

and the Hermite basis function for  $\mathcal{M}_1^{3,0}$  be

$$\{\Psi_n\}_{n=1}^{2N} = \{v_0, v_1, \dots, v_{N-1}, v_N, s_1, \dots, s_{N-1}\},$$

where  $v_i$ ,  $i = 0, 1, 2, \dots, N$  are value functions and  $s_j$ ,  $j = 1, 2, \dots, N - 1$  are slope functions for Hermite cubic basis functions. In this context, see [31] for the expressions of  $v_i$  and  $s_j$ .

With the help of the above defined basis functions, the collocation approximation is expressed in the form

$$U(x, y) = \sum_{m=1}^{4N+2} \sum_{n=1}^{2N} U_{m,n} \Phi_m(x) \Psi_n(y). \quad (2.2)$$

Substituting Eq. (2.2) into Eq. (2.1) leads to the following structure of the collocation equations:

$$(A_1 \otimes B_2 + B_1 \otimes A_2) u = F, \quad (2.3)$$

where

$$\begin{aligned}A_1 &= \left[ a_{i,j}^{(1)} \right]_{i,j=1}^{4N+2}, \quad a_{i,j}^{(1)} = \Phi_j''(\xi_i) + w^2 \Phi_j(\xi_i), \\ A_2 &= \left[ a_{i,j}^{(2)} \right]_{i,j=1}^{2N}, \quad a_{i,j}^{(2)} = \Psi_j''(\xi_i), \\ B_1 &= \left[ b_{i,j}^{(1)} \right]_{i,j=1}^{4N+2}, \quad b_{i,j}^{(1)} = \Phi_j(\xi_i), \\ B_2 &= \left[ b_{i,j}^{(2)} \right]_{i,j=1}^{2N}, \quad b_{i,j}^{(2)} = \Psi_j(\xi_i).\end{aligned}$$

To solve (2.3),  $A_2$  and  $B_2$  are formulated into the generalized eigenvalues and corresponding eigenfunctions as follows.

$$A_2 \varphi = \lambda B_2 \varphi. \quad (2.4)$$



As the matrices  $A_2$  and  $B_2$  contain the second derivative of the Hermite basis  $\Psi$ , the following classical eigenvalue problem is considered,

$$u''(y) = \lambda u(y), \quad 0 < y < 1, \quad (2.5)$$

$$u'(0) = u'(1) = 0. \quad (2.6)$$

We note that the generalized eigenvalues and corresponding eigenfunctions of (2.4) give the Hermite cubic collocation approximations to the eigenvalues and corresponding eigenfunctions of (2.5)-(2.6). It is worth mentioning that these values can be obtained by using the Eqs. (3.2.1 – 3.2.4), (3.2.13 – 3.2.16), and Appendix of [11].

Therefore, the  $2N$  generalized eigenvalues of the Equations (2.5)-(2.6) are given by

$$\lambda_0^- = 0, \quad \lambda_N^- = -\frac{9}{h^2},$$

$$\lambda_j^\pm = -\frac{12}{h^2} \left( \frac{8 + \eta_j \pm \mu_j}{7 - \eta_j} \right), \quad j = 1, 2, \dots, N-1,$$

where

$$\eta_j = \cos\left(\frac{j\pi}{N}\right), \quad \mu_j = \sqrt{43 + 40\eta_j - 2\eta_j^2}.$$

As the system (2.3) is a two-dimensional problem and to reduce it to a set of independent one-dimensional problems, two real non-singular matrices  $\mathcal{Z}$  and  $\Lambda$  are required.

Hence, from Appendix [11], we now define

$$\mathcal{Z} = 3\sqrt{3} \left[ \begin{array}{c|c} C\Lambda_\alpha^- & \tilde{C}\Lambda_\alpha^+ \\ \hline \mathbf{0} | -S\Lambda_\beta^- & \mathbf{0} | -S\Lambda_\beta^+ \end{array} \right], \quad (2.7)$$

and

$$\Lambda = \text{diag}(\lambda_0^-, \lambda_1^-, \dots, \lambda_{N-1}^-, \lambda_N^-, \lambda_1^+, \dots, \lambda_{N-1}^+), \quad (2.8)$$

where

$$\mathcal{S} = \sqrt{\frac{2}{N}} \sin\left(\frac{mn\pi}{N}\right)_{m,n=1}^{N-1}, \quad \tilde{C} = \sqrt{\frac{2}{N}} \cos\left(\frac{mn\pi}{N}\right)_{m=0,n=1}^{N,N-1}, \quad C = \sqrt{\frac{2}{N}} \cos\left(\frac{mn\pi}{N}\right)_{m=0,n=0}^N,$$

$$\Lambda_\alpha^\pm = \text{diag}(\alpha_1^\pm, \alpha_2^\pm, \dots, \alpha_{N-1}^\pm), \quad \Lambda_\beta^- = \text{diag}(\beta_1^-, \beta_2^-, \dots, \beta_{N-1}^-), \quad \Lambda_\beta^+ = \text{diag}\left(1, \beta_1^+, \beta_2^+, \dots, \beta_{N-1}^+, \frac{1}{\sqrt{3}}\right),$$

$$\alpha_j^\pm = (5 + 4\eta_j \mp \mu_j)\nu_j^\pm, \quad \beta_j^\pm = 18 \sin\left(\frac{j\pi}{N}\right)\nu_j^\pm,$$

$$\nu_j^\pm = \left[ 27(1 + \eta_j)(8 + \eta_j \mp \mu_j)^2 + (1 - \eta_j)(11 + 7\eta_j \mp 4\mu_j)^2 \right]^{-\frac{1}{2}}.$$

Next, we have discussed the method of finding the solution of (2.3) using the matrices  $\mathcal{Z}$  and  $\Lambda$ .

### 3. MATRIX DECOMPOSITION ALGORITHM

In this section, a matrix decomposition algorithm (MDA) is implemented to convert the two-dimensional problem to a set of one-dimensional problems. The detailed application of this algorithm for two dimensional elliptic boundary value problems can be found in the following articles [6, 7, 9, 10, 16, 25]. Henceforth, the matrices  $\mathcal{Z}$  and  $\Lambda$  of the previous section are used to decompose the matrix system (2.3) as follows.

Pre-multiply the matrix  $(I \otimes \mathcal{Z}^T B_2^T)$  to Equation (2.3), we get

$$(I \otimes \mathcal{Z}^T B_2^T) (A_1 \otimes B_2 + B_1 \otimes A_2) u = (I \otimes \mathcal{Z}^T B_2^T) F. \quad (3.1)$$

Further, the expression (3.1) is rewritten as

$$(I \otimes \mathcal{Z}^T B_2^T) (A_1 \otimes B_2 + B_1 \otimes A_2) (I \otimes \mathcal{Z}) (I \otimes \mathcal{Z}^{-1}) u = (I \otimes \mathcal{Z}^T B_2^T) F. \quad (3.2)$$



Now the left-hand side of the expression (3.2) can be simplified as

$$(A_1 \otimes I + B_1 \otimes \Lambda) (I \otimes \mathcal{Z}^{-1}) u = (I \otimes \mathcal{Z}^T B_2^T) F. \quad (3.3)$$

Note that in (3.3), the following facts are used:

$$\mathcal{Z}^T B_2^T A_2 \mathcal{Z} = \Lambda, \quad \text{and} \quad \mathcal{Z}^T B_2^T B_2 \mathcal{Z} = I.$$

Assuming that  $(I \otimes \mathcal{Z}^{-1}) u = v$  and  $(I \otimes \mathcal{Z}^T B_2^T) F = g$ , then the expression (3.3) becomes

$$(A_1 \otimes I + B_1 \otimes \Lambda) v = g. \quad (3.4)$$

Now the matrix system (3.4) can be solved by following the steps mentioned below.

**Step 1.** Compute  $g = (I \otimes \mathcal{Z}^T B_2^T) F$ .

**Step 2.** Solve  $(A_1 \otimes I + B_1 \otimes \Lambda) v = g$ .

**Step 3.** Compute the approximate solution  $u = (I \otimes \mathcal{Z}) v$ .

By the definition of Kronecker product, the system defined in (3.4) can be simplified in the form

$$(A_1 + \lambda_i B_1) u_{ij} = g_{ij}, \quad i = 1, 2, \dots, 2N, \quad j = 1, 2, \dots, 4N + 2. \quad (3.5)$$

Suppose  $E_i = A_1 + \lambda_i B_1$ , for  $i = 1, 2, \dots, 2N$ . Now the system (3.5) becomes

$$E_i u_{ij} = g_{ij}, \quad i = 1, 2, \dots, 2N, \quad j = 1, 2, \dots, 4N + 2. \quad (3.6)$$

The matrix form of the system (3.6) is

$$\begin{bmatrix} L_b & & & & & & & & \\ S_1 & T_1 & & & & & & & \\ -C_1 & -D_1 & I_2 & & & & & & \\ & & & \ddots & & & & & \\ & & & S_i & T_i & & & & \\ & & & -C_i & -D_i & I_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & S_N & T_N & \\ & & & & & & -C_N & -D_N & I_2 \\ & & & & & & & & R_b \end{bmatrix} \begin{bmatrix} u_{0,i} \\ v_{0,i} \\ u_{1,i} \\ \vdots \\ u_{j-1,i} \\ v_{j-1,i} \\ \vdots \\ u_{N-1,i} \\ v_{N-1,i} \\ u_{N,i} \end{bmatrix} = \begin{bmatrix} g_0 \\ f_1 \\ \mathbf{0} \\ \vdots \\ f_i \\ \mathbf{d} \\ \vdots \\ f_N \\ \mathbf{0} \\ g_1 \end{bmatrix}, \quad (3.7)$$

where  $L_b = [1 \ 0]$  and  $R_b = [1 \ 0]$ , arise from the left and right boundary conditions of (1.2), respectively. Furthermore,  $u_{ji} = [u_{j1,i} \ u_{j2,i}]^T$ ,  $v_{ji} = [v_{j1,i} \ v_{j2,i}]^T$  for  $i = 1, 2, \dots, 2N$ , and  $I_2$  is the identity matrix of order 2. In (3.7),  $\mathbf{d} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$  is obtained from (1.4). In connection with (3.7), the entries of the matrices  $S_i$  and  $T_i$  on the intervals  $[x_0, x_1]$ ,  $[x_1, x_2]$ ,  $\dots$ ,  $[x_{i-1}, x_i]$  are

$$S_i = \begin{bmatrix} \omega_-^2 + \lambda_i & \frac{1}{2}(\lambda_i + 1)\theta_1 h \\ \omega_-^2 + \lambda_i & \frac{1}{2}(\lambda_i + 1)\theta_2 h \end{bmatrix},$$

$$T_i = \begin{bmatrix} 2 + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^2 & 3\theta_1 h + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^3 \\ 2 + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^2 & 3\theta_2 h + (\omega_-^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^3 \end{bmatrix},$$

and on the intervals  $[x_i, x_{i+1}]$ ,  $[x_{i+1}, x_{i+2}]$ ,  $\dots$ ,  $[x_{N-1}, x_N]$

$$S_i = \begin{bmatrix} (\omega_+^2 + \lambda_i) & \frac{1}{2}\theta_1 h \\ (\omega_+^2 + \lambda_i) & \frac{1}{2}\theta_2 h \end{bmatrix},$$

$$T_i = \begin{bmatrix} 2 + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^2 & 3\theta_1 h + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_1 h)^3 \\ 2 + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^2 & 3\theta_2 h + (\omega_+^2 + \lambda_i)(\frac{1}{2}\theta_2 h)^3 \end{bmatrix},$$

where  $\theta_1 = 1 + \rho_1$  and  $\theta_2 = 1 + \rho_2$ .



It can be observed that for every value of  $h = \frac{1}{N}$  with different  $\theta_1$  and  $\theta_2$ , the columns of matrices  $T_i$ ,  $i = 1, 2, \dots, N$  are linearly independent. Therefore, the matrices  $T_i$  are non-singular. In the next theorem, it is proved that the system (3.7) has one and only one solution.

**Theorem 3.1.** *Let  $T_i$ ,  $i = 1, 2, \dots, N$  be non-singular matrices. Then the almost block diagonal linear system of order  $4N + 2$  defined in (3.7) has a unique solution.*

*Proof.* Given that the matrices  $T_i$  are non-singular for  $i = 1, 2, \dots, N$ , to prove that the coefficient matrix defined in (3.7) is non-singular, it is sufficient to show that there exists a trivial solution to the homogeneous system (3.7). Hence from (3.7), we obtain

$$L_b u_{0,i} = 0, \quad (3.8)$$

$$S_1 u_{0,i} + T_1 v_{0,i} = 0, \quad (3.9)$$

$$-C_1 u_{0,i} - D_1 v_{0,i} + I_2 u_{1,i} = 0. \quad (3.10)$$

From (3.8), we have  $u_{0,i} = 0$ . Further, substituting  $u_{0,i} = 0$  in (3.9), we get

$$T_1 v_{0,i} = 0, \quad (3.11)$$

and since  $T_1$  is non-singular implies that  $v_{0,i} = 0$ . Again substituting  $u_{0,i}$  and  $v_{0,i}$  in (3.10) gives  $u_{1,i} = 0$ . Since each  $T_i$  is non-singular, solve the  $i^{th}$  row equation of (3.7) as follows

$$S_i u_{j-1,i} + T_i v_{j-1,i} = 0, \quad (3.12)$$

$$-C_i u_{j-1,i} - D_i v_{j-1,i} + I_2 u_{j,i} = 0. \quad (3.13)$$

Similar to the above analysis, it is conclusive that  $u_{j-1,i} = v_{j-1,i} = u_{j,i} = 0$  for  $i = 1, 2, \dots, N$ . Hence, the only solution that exists is trivial. Therefore, the coefficient matrix in (3.7) is invertible and has a unique solution.  $\square$

#### 4. NUMERICAL RESULTS

In this section, the numerical implementation of the proposed method is demonstrated to show the accuracy and order of convergence of Helmholtz interface problems that involve different forms of Neumann boundary conditions. Furthermore, we have shown that the order of convergence of the proposed scheme is better (for Neumann boundary conditions) than the order of convergence of discontinuous problems obtained using the finite difference scheme mentioned in [18, 20, 22, 23]. The performance of this method is computed using  $L^\infty$ ,  $L^2$ , and  $H^1$  norms, where  $L^\infty$  error is estimated by determining the maximum absolute error at  $10 \times 10$  equally spaced points in each sub-interval  $I_j \times I_j$ ,  $j = 1, \dots, N$ . To estimate the  $L^2$  and  $H^1$  error, the composite three-point Gauss quadrature is used. It is assumed that  $u_h$  is the approximate solution with step size  $h$  corresponding to the exact solution  $u(x, y)$ . Furthermore, the maximum absolute error and the first derivatives at the nodes are computed using the  $\ell^\infty$  norm. It is noted that the input  $N$  generates the matrices of size  $(4N + 2) \times 2N$  of the required order. In the following examples, the consideration of degree  $r$  piecewise polynomials in OSCM ensures the spatial accuracy of  $r + 1$  in the  $L^\infty$  and  $L^2$  norms, and  $r$  in  $H^1$  norm. We also expect  $2r - 2$  order superconvergence in the  $\ell^\infty$  norm of the approximation and its first spatial derivative.

In the following example, Helmholtz interface problem is considered with piecewise constant wave numbers and hence computed the error bounds along with order of convergence.

**Example 4.1.** The two-dimensional interface problem under consideration is

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) + w^2 u(x, y) &= f(x, y), \quad (x, y) \in [0, 1] \times [0, 1], \\ u_x(0, y) &= 0, \quad u_x(1, y) = 1, \text{ on } \partial\Omega, \\ u_y(x, 0) &= 0, \quad u_y(x, 1) = 0, \text{ on } \partial\Omega, \end{aligned}$$

where

$$w^2(x) = \begin{cases} w_-^2 & , \quad x \in [0, 0.5] \times [0, 1], \\ w_+^2 & , \quad x \in (0.5, 1] \times [0, 1]. \end{cases}$$



TABLE 1. Order of convergence and error using  $\ell^\infty$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u - u_h\ _{\ell^\infty}$	order	error $\ u - u_h\ _{\ell^\infty}$	order
4	$7.6926 \times 10^{-4}$		$1.4989 \times 10^{-3}$	
8	$4.7442 \times 10^{-5}$	4.0192	$9.4696 \times 10^{-5}$	3.9845
12	$9.3467 \times 10^{-6}$	4.0065	$1.8757 \times 10^{-5}$	3.9932
16	$2.9546 \times 10^{-6}$	4.0032	$5.9411 \times 10^{-6}$	3.9963
20	$1.2097 \times 10^{-6}$	4.0019	$2.4347 \times 10^{-6}$	3.9977
24	$5.8323 \times 10^{-7}$	4.0013	$1.1745 \times 10^{-6}$	3.9984

TABLE 2. Order of convergence and error using  $L^2$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u - u_h\ _{L^2}$	order	error $\ u - u_h\ _{L^2}$	order
4	$1.0134 \times 10^{-3}$		$1.1293 \times 10^{-3}$	
8	$6.2114 \times 10^{-5}$	4.0281	$6.9111 \times 10^{-5}$	4.0303
12	$1.2224 \times 10^{-5}$	4.0091	$1.3638 \times 10^{-5}$	4.0024
16	$3.8628 \times 10^{-6}$	4.0045	$4.3147 \times 10^{-6}$	4.0005
20	$1.5813 \times 10^{-6}$	4.0027	$1.7673 \times 10^{-6}$	4.0001
24	$7.6232 \times 10^{-7}$	4.0018	$8.5227 \times 10^{-7}$	4.0000

The exact solution is given by,

$$u(x, y) = \begin{cases} \cos(\pi x) \cos(\pi y), & (x, y) \in \Omega^-, \\ x + \cos(\pi x) \cos(\pi y), & (x, y) \in \Omega^+, \end{cases}$$

and the interface conditions at  $x = \frac{1}{2}$  are

$$[u] = -\frac{1}{2}, \quad [u_x] = 1.$$

The source function  $f(x, y)$  is computed with the help of the exact solution, i.e.,

$$f(x, y) = \begin{cases} (-2\pi^2 + w_-^2) \cos(\pi x) \cos(\pi y), & (x, y) \in \Omega^-, \\ (-2\pi^2 + w_+^2) \cos(\pi x) \cos(\pi y) + w_+^2 x, & (x, y) \in \Omega^+. \end{cases}$$

We have computed the approximate solutions by considering the following wave numbers:  $w_-^2 = 1, w_+^2 = 9$  and  $w_-^2 = 25, w_+^2 = 100$ . The size of the matrices versus the order of convergence and the error **concerning**  $\ell^\infty$ ,  $L^2$ ,  $L^\infty$ , and  $H^1$  norms are presented in the Tables 1-4. It is observed that when the matrices size increases, this method confers the expected order of convergence, which is close to 4.

For the super-convergence result, we have calculated the partial derivatives of the solution with respect to  $x, y$ , and  $xy$  and thereafter its error is elaborated in the Tables 5-7. Generally, at the nodal points, the order of convergence of solution derivatives reduces the actual order of convergence by 1 while OSC method for this problem gives the actual order of convergence that appears in the Tables 5-7.



TABLE 3. Order of convergence and error using  $L^\infty$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u - u_h\ _{L^\infty}$	order	error $\ u - u_h\ _{L^\infty}$	order
4	$1.3441 \times 10^{-3}$		$1.2855 \times 10^{-3}$	
8	$9.2106 \times 10^{-5}$	3.8672	$8.1217 \times 10^{-5}$	3.9844
12	$1.8500 \times 10^{-5}$	3.9589	$1.6088 \times 10^{-5}$	3.9930
16	$5.8875 \times 10^{-6}$	3.9798	$5.0960 \times 10^{-6}$	3.9961
20	$2.4180 \times 10^{-6}$	3.9880	$2.0885 \times 10^{-6}$	3.9976
24	$1.1678 \times 10^{-6}$	3.9920	$1.0075 \times 10^{-6}$	3.9984

TABLE 4. Order of convergence and error using  $H^1$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u - u_h\ _{H^1}$	order	error $\ u - u_h\ _{H^1}$	order
4	$1.2322 \times 10^{-2}$		$1.2702 \times 10^{-2}$	
8	$1.5131 \times 10^{-3}$	3.0256	$1.5171 \times 10^{-3}$	3.0657
12	$4.4684 \times 10^{-4}$	3.0082	$4.4723 \times 10^{-4}$	3.0125
16	$1.8829 \times 10^{-4}$	3.0041	$1.8837 \times 10^{-4}$	3.0056
20	$9.6353 \times 10^{-5}$	3.0024	$9.6378 \times 10^{-5}$	3.0032
24	$5.5744 \times 10^{-5}$	3.0016	$5.5753 \times 10^{-5}$	3.0021

TABLE 5. Order of convergence and error of  $u_x$  using  $\ell^\infty$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u_x - u_{hx}\ _{\ell^\infty}$	order	error $\ u_x - u_{hx}\ _{\ell^\infty}$	order
4	$7.2882 \times 10^{-4}$		$3.4461 \times 10^{-3}$	
8	$4.7485 \times 10^{-5}$	3.9400	$2.2542 \times 10^{-4}$	3.9343
12	$9.3863 \times 10^{-6}$	3.9983	$3.9770 \times 10^{-5}$	3.9882
16	$2.9418 \times 10^{-6}$	4.0331	$1.4269 \times 10^{-5}$	3.9882
20	$1.2130 \times 10^{-6}$	3.9703	$5.8541 \times 10^{-6}$	3.9928
24	$5.8436 \times 10^{-7}$	4.0056	$2.8256 \times 10^{-6}$	3.9952

In the next example, the wave numbers are defined at two interface conditions and the order of convergence, and error with regard to different norms are evaluated.

**Example 4.2.** We consider the following two-dimensional interface problem

$$\begin{aligned}
 u_{xx}(x, y) + u_{yy}(x, y) + w^2 u(x, y) &= f(x, y), \quad (x, y) \in [0, 1] \times [0, 1], \\
 u(0, y) &= 0, \quad u(1, y) = 0, \text{ on } \partial\Omega, \\
 u_y(x, 0) &= 0, \quad u_y(x, 1) = 0, \text{ on } \partial\Omega,
 \end{aligned}$$





TABLE 6. Order of convergence and error of  $u_y$  using  $\ell^\infty$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u_y - u_{hy}\ _{\ell^\infty}$	order	error $\ u_y - u_{hy}\ _{\ell^\infty}$	order
4	$1.3472 \times 10^{-3}$		$3.4403 \times 10^{-3}$	
8	$8.0430 \times 10^{-5}$	4.0661	$2.2595 \times 10^{-4}$	3.9285
12	$1.5748 \times 10^{-5}$	4.0218	$4.5055 \times 10^{-5}$	3.9767
16	$4.9673 \times 10^{-6}$	4.0108	$1.4304 \times 10^{-5}$	3.9882
20	$2.0316 \times 10^{-6}$	4.0065	$5.8685 \times 10^{-6}$	3.9929
24	$9.7900 \times 10^{-7}$	4.0043	$2.8325 \times 10^{-6}$	3.9952

TABLE 7. Order of convergence and error of  $u_{xy}$  using  $\ell^\infty$  norm.

$N$	$w_-^2 = 1$	$w_+^2 = 9$	$w_-^2 = 25$	$w_+^2 = 100$
	error $\ u_{xy} - u_{hxy}\ _{\ell^\infty}$	order	error $\ u_{xy} - u_{hxy}\ _{\ell^\infty}$	order
4	$1.1690 \times 10^{-3}$		$7.9871 \times 10^{-3}$	
8	$7.8204 \times 10^{-5}$	3.9018	$4.6827 \times 10^{-4}$	4.0922
12	$1.5633 \times 10^{-5}$	3.9706	$9.6344 \times 10^{-5}$	3.8995
16	$4.9733 \times 10^{-6}$	3.9811	$3.0906 \times 10^{-5}$	3.9522
20	$2.0460 \times 10^{-6}$	3.9803	$1.2739 \times 10^{-5}$	3.9718
24	$9.8862 \times 10^{-7}$	3.9894	$6.1643 \times 10^{-6}$	3.9814

where

$$w^2(x) = \begin{cases} w_1^2, & x \in \Omega_1 = [0, 0.25] \times [0, 1], \\ w_2^2, & x \in \Omega_2 = (0.25, 0.75] \times [0, 1], \\ w_3^2, & x \in \Omega_3 = (0.75, 1] \times [0, 1]. \end{cases}$$

The exact solution is

$$u(x, y) = \begin{cases} \sin(\pi x) \cos(\pi y), & (x, y) \in \Omega_1, \\ x + \sin(\pi x) \cos(\pi y), & (x, y) \in \Omega_2, \\ 1 - x + \sin(\pi x) \cos(\pi y), & (x, y) \in \Omega_3, \end{cases}$$

and the interface conditions at  $x = \frac{1}{2}$  and at  $x = \frac{3}{4}$  are

$$[u] = \frac{1}{4}, \quad [u_x] = 1,$$

$$[u] = -\frac{1}{2}, \quad [u_x] = -2,$$

respectively. The source function  $f(x, y)$  is calculated with the help of the exact solution as

$$f(x, y) = \begin{cases} (-2\pi^2 + w_1^2) \sin(\pi x) \cos(\pi y), & (x, y) \in \Omega_1, \\ (-2\pi^2 + w_2^2) \cos(\pi x) \cos(\pi y) + w_2^2 x, & (x, y) \in \Omega_2, \\ (-2\pi^2 + w_3^2) \cos(\pi x) \cos(\pi y) + w_3^2(1 - x), & (x, y) \in \Omega_3. \end{cases}$$



TABLE 8. Order of convergence and error using  $\ell^\infty$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error			error		
	$\ u - u_h\ _{\ell^\infty}$			$\ u - u_h\ _{\ell^\infty}$		
	order			order		
4	$6.9359 \times 10^{-4}$			$7.8240 \times 10^{-4}$		
8	$4.2944 \times 10^{-5}$			$5.2043 \times 10^{-5}$		
12	$8.4669 \times 10^{-6}$			$1.0363 \times 10^{-5}$		
16	$2.6772 \times 10^{-6}$			$3.2878 \times 10^{-6}$		
20	$1.0962 \times 10^{-6}$			$1.3499 \times 10^{-6}$		
24	$5.2857 \times 10^{-7}$			$6.5076 \times 10^{-7}$		

TABLE 9. Order of convergence and error using  $L^2$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error			error		
	$\ u - u_h\ _{L^2}$			$\ u - u_h\ _{L^2}$		
	order			order		
4	$9.0102 \times 10^{-4}$			$8.8622 \times 10^{-4}$		
8	$5.5270 \times 10^{-5}$			$5.4318 \times 10^{-5}$		
12	$1.0878 \times 10^{-5}$			$1.0709 \times 10^{-5}$		
16	$3.4373 \times 10^{-6}$			$3.3866 \times 10^{-6}$		
20	$1.4071 \times 10^{-6}$			$1.3869 \times 10^{-6}$		
24	$6.7833 \times 10^{-7}$			$6.6873 \times 10^{-7}$		

TABLE 10. Order of convergence and error using  $L^\infty$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error			error		
	$\ u - u_h\ _{L^\infty}$			$\ u - u_h\ _{L^\infty}$		
	order			order		
4	$1.1068 \times 10^{-3}$			$1.0655 \times 10^{-3}$		
8	$7.6518 \times 10^{-5}$			$7.0895 \times 10^{-5}$		
12	$1.5390 \times 10^{-5}$			$1.4092 \times 10^{-5}$		
16	$4.9000 \times 10^{-6}$			$4.4610 \times 10^{-6}$		
20	$2.0218 \times 10^{-6}$			$1.8262 \times 10^{-6}$		
24	$9.7221 \times 10^{-7}$			$8.8007 \times 10^{-7}$		

In the following Tables: 8–11, the error and order of convergence are computed for the wave numbers  $\{w_-^2 = 1, w_+^2 = 2, w_3^2 = 3\}$  and  $\{w_1^2 = 1, w_2^2 = 10, w_+^2 = 100\}$ . It is also noticed that the fourth order of convergence is obtained even if the size of the matrix is 144. Thereafter, we have presented the error and super-convergence result of the partial derivatives of the solution corresponding to  $x$ ,  $y$ , and  $xy$  that are shown in the Tables: 12–14. Additionally, we have plotted the approximate and exact solution graph in Figures 1 and 2.

The following figures are the approximate solution and exact solution plot of Example 4.2 for  $N = 24$  which gives the matrix of order 4704. The values of  $(x_i, y_i)$  vary at the nodal points and the figure plotted corresponding to its



TABLE 11. Order of convergence and error using  $H^1$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error $\ u - u_h\ _{H^1}$			error $\ u - u_h\ _{H^1}$		
4	$1.2229 \times 10^{-2}$			$1.2379 \times 10^{-2}$		
8	$1.5104 \times 10^{-3}$		3.0173	$1.5123 \times 10^{-3}$		3.0331
12	$4.4650 \times 10^{-4}$		3.0058	$4.4668 \times 10^{-4}$		3.0077
16	$1.8821 \times 10^{-4}$		3.0029	$1.8825 \times 10^{-4}$		3.0036
20	$9.6327 \times 10^{-5}$		3.0017	$9.6339 \times 10^{-5}$		3.0021
24	$5.5733 \times 10^{-5}$		3.0011	$5.5738 \times 10^{-5}$		3.0014

TABLE 12. Order of convergence and error of  $u_x$  using  $\ell^\infty$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error $\ u_x - u_{hx}\ _{\ell^\infty}$			error $\ u_x - u_{hx}\ _{\ell^\infty}$		
4	$1.1170 \times 10^{-3}$			$1.8858 \times 10^{-3}$		
8	$6.6909 \times 10^{-5}$		4.0613	$2.0993 \times 10^{-4}$		3.1672
12	$1.3109 \times 10^{-5}$		4.0202	$4.4329 \times 10^{-5}$		3.8354
16	$4.1357 \times 10^{-6}$		4.0100	$1.4335 \times 10^{-5}$		3.9242
20	$1.6917 \times 10^{-6}$		4.0060	$5.9297 \times 10^{-6}$		3.9559
24	$8.1525 \times 10^{-7}$		4.0040	$2.8748 \times 10^{-6}$		3.9710

TABLE 13. Order of convergence and error of  $u_y$  using  $\ell^\infty$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error $\ u_y - u_{hy}\ _{\ell^\infty}$			error $\ u_y - u_{hy}\ _{\ell^\infty}$		
4	$1.1017 \times 10^{-3}$			$1.3105 \times 10^{-3}$		
8	$6.6180 \times 10^{-5}$		4.0572	$1.0064 \times 10^{-4}$		3.7028
12	$1.2973 \times 10^{-5}$		4.0188	$2.0654 \times 10^{-5}$		3.9058
16	$4.0938 \times 10^{-6}$		4.0093	$6.6217 \times 10^{-6}$		3.9542
20	$1.6747 \times 10^{-6}$		4.0056	$2.7288 \times 10^{-6}$		3.9728
24	$8.0710 \times 10^{-7}$		4.0037	$1.3203 \times 10^{-6}$		3.9819

$u_h(x_i, y_i)$  and  $u(x_i, y_i)$ . These graphs signify that although there are discontinuities in the interface problem still the approximate solution is almost equal to the exact solution.

In the last example, the Helmholtz interface problem is solved with a piecewise continuous wave function and the order of convergence followed by the errors is presented.

**Example 4.3.** Let us assume the two-dimensional interface problem

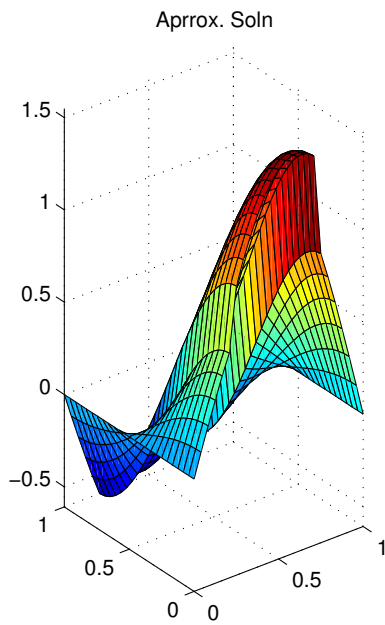
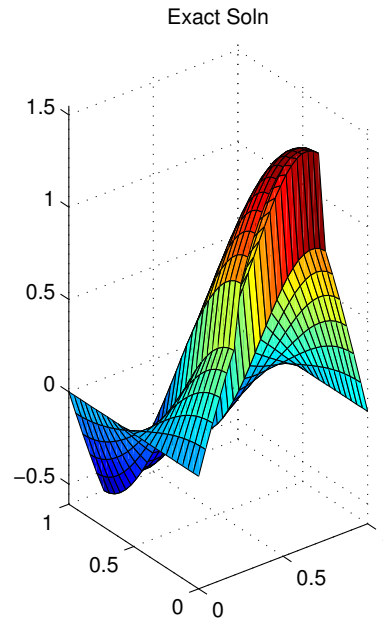
$$u_{xx}(x, y) + u_{yy}(x, y) + w^2 u(x, y) = f(x, y), \quad (x, y) \in [0, 1] \times [0, 1],$$

$$u_x(0, y) = 0, \quad u_x(1, y) = 1, \quad \text{on } \partial\Omega,$$



TABLE 14. Order of convergence and error of  $u_{xy}$  using  $\ell^\infty$  norm.

$N$	$w_1^2 = 1$	$w_2^2 = 2$	$w_3^2 = 3$	$w_1^2 = 1$	$w_2^2 = 10$	$w_+^2 = 100$
	error			error		
	$\ u_{xy} - u_{hxy}\ _{\ell^\infty}$		order	$\ u_{xy} - u_{hxy}\ _{\ell^\infty}$		order
4	$1.4785 \times 10^{-4}$			$1.7331 \times 10^{-3}$		
8	$1.0731 \times 10^{-5}$		3.7843	$4.0482 \times 10^{-4}$		2.0980
12	$2.5180 \times 10^{-6}$		3.5754	$9.3134 \times 10^{-5}$		3.6240
16	$8.4065 \times 10^{-7}$		3.8134	$3.0888 \times 10^{-5}$		3.8364
20	$3.5265 \times 10^{-7}$		3.8930	$1.2918 \times 10^{-5}$		3.9066
24	$1.7225 \times 10^{-7}$		3.9302	$6.2992 \times 10^{-6}$		3.9392

FIGURE 1. The plot of approximate solution for  $N = 24$ .FIGURE 2. The plot of the exact solution for  $N = 24$ .

$$u_y(x, 0) = 0, \quad u_y(x, 1) = 0, \quad \text{on } \partial\Omega,$$

where

$$w^2(x) = \begin{cases} w_-^2 = x + 1, & x \in [0, 0.5] \times [0, 1], \\ w_+^2 = x + 2, & x \in (0.5, 1] \times [0, 1]. \end{cases}$$

The exact solution is given by

$$u(x, y) = \begin{cases} \cos(\pi x) \cos(\pi y), & (x, y) \in \Omega^- \\ x + \cos(\pi x) \cos(\pi y), & (x, y) \in \Omega^+, \end{cases}$$



TABLE 15. Order of convergence and error using  $\ell^\infty$  and  $L^\infty$  norm.

$N$	$w_-^2 = x + 1$	$w_+^2 = x + 2$	$w_-^2 = x + 1$	$w_+^2 = x + 2$
	error $\ u - u_h\ _{\ell^\infty}$	order	error $\ u - u_h\ _{L^\infty}$	order
4	$7.0745 \times 10^{-4}$		$1.1208 \times 10^{-3}$	
8	$4.3823 \times 10^{-5}$	4.0129	$7.7459 \times 10^{-5}$	3.8550
12	$8.6405 \times 10^{-6}$	4.0045	$1.5579 \times 10^{-5}$	3.9555
16	$2.7381 \times 10^{-6}$	4.0023	$4.9602 \times 10^{-6}$	3.9782
20	$1.1187 \times 10^{-6}$	4.0014	$2.0376 \times 10^{-6}$	3.9870
24	$5.3942 \times 10^{-7}$	4.0009	$9.8412 \times 10^{-7}$	3.9914

TABLE 16. Order of convergence and error using  $L^2$  and  $H^1$  norm.

$N$	$w_-^2 = x + 1$	$w_+^2 = x + 2$	$w_-^2 = x + 1$	$w_+^2 = x + 2$
	error $\ u - u_h\ _{L^2}$	order	error $\ u - u_h\ _{H^1}$	order
4	$9.0184 \times 10^{-4}$		$1.2230 \times 10^{-2}$	
8	$5.5321 \times 10^{-5}$	4.0270	$1.5105 \times 10^{-3}$	3.0173
12	$1.0888 \times 10^{-5}$	4.0090	$4.4650 \times 10^{-4}$	3.0058
16	$3.4404 \times 10^{-6}$	4.0045	$1.8821 \times 10^{-4}$	3.0029
20	$1.4084 \times 10^{-6}$	4.0027	$9.6327 \times 10^{-5}$	3.0017
24	$6.7896 \times 10^{-7}$	4.0018	$5.5733 \times 10^{-5}$	3.0011

and the interface conditions at  $x = \frac{1}{2}$  are

$$[u] = -\frac{1}{2}, \quad [u_x] = 1.$$

The source function  $f(x, y)$  is determined by the help of the exact solution as

$$f(x, y) = \begin{cases} (-2\pi^2 + w_-^2) \cos(\pi x) \cos(\pi y), & (x, y) \in \Omega^-, \\ (-2\pi^2 + w_+^2) \cos(\pi x) \cos(\pi y) + w_+^2 x, & (x, y) \in \Omega^+. \end{cases}$$

We have calculated the estimate solutions by assuming the wave functions as  $w_-^2 = x + 1, w_+^2 = x + 2$  and  $w_-^2 = x + 1, w_+^2 = x + 2$ . Further, the computation of order of convergence and error concerning various norms is presented in the Tables 15-17.

## 5. CONCLUSION

This article focuses on implementing the MDA algorithm for solving the two-dimensional Helmholtz interface problem with Neumann boundary conditions. The MDA algorithm effectively resolves the problem, and we computed the approximate solution for three different types of wave functions. This method remains effective even in scenarios involving large wave numbers. The associated errors were presented using various norms. Moreover, we observed the desired order of convergence and achieved superconvergence results at nodal points, even with a small matrix size. In the future, the OSC method for solving Helmholtz interface problems can be extended to the annulus region and to a disc. A theoretical convergence analysis for Helmholtz problems with interfaces and the extension of the OSC method to Robbin boundary conditions are topics of future research.



TABLE 17. Order of convergence and error of  $u_x$ ,  $u_y$  and  $u_{xy}$  using  $\ell^\infty$  norm.

$N$	$w_-^2 = x + 1$		$w_+^2 = 2 + x$			
	$\ u_x - u_{hx}\ _{\ell^\infty}$	order	$\ u_y - u_{hy}\ _{\ell^\infty}$	order	$\ u_{xy} - u_{hxy}\ _{\ell^\infty}$	order
4	$1.0987 \times 10^{-3}$		$1.1527 \times 10^{-3}$		$1.0878 \times 10^{-4}$	
8	$6.5998 \times 10^{-5}$	4.0572	$6.9056 \times 10^{-5}$	4.0611	$9.8381 \times 10^{-6}$	3.4669
12	$1.2938 \times 10^{-5}$	4.0188	$1.3529 \times 10^{-5}$	4.0203	$2.3422 \times 10^{-6}$	3.5396
16	$4.0826 \times 10^{-6}$	4.0093	$4.2683 \times 10^{-6}$	4.0101	$7.8312 \times 10^{-7}$	3.8083
20	$1.6701 \times 10^{-6}$	4.0056	$1.7459 \times 10^{-6}$	4.0060	$3.3128 \times 10^{-7}$	3.8555
24	$8.0488 \times 10^{-7}$	4.0037	$8.4136 \times 10^{-7}$	4.0040	$1.6201 \times 10^{-7}$	3.9232

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