Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 4, 2025, pp. 1336-1356
DOI:10.22034/cmde.2024.62097.2714

Computational applications by ITEM and the variational method for solving the Hamiltonian amplitude equation

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Abstract

The paper presents a significant improvement to the implementation of the improved $\tan(\phi(\xi)/2)$-expansion method (ITEM) for solving the Hamiltonian amplitude equation (HAE). We seek to improve the exact solutions by applying the ITEM. Computed solutions are compared with previously published results obtained using the simplest equation method [15] and the $(G'/G, 1/G)$-expansion method [13]. There is clear evidence that the new approach produces results that are as good as, if not better than, published results determined using the other methods. The main advantage of the method is that it offers further solutions. By using this method, exact solutions, including the hyperbolic function solution, traveling wave solution, soliton solution, rational function solution, and periodic wave solution of this equation, have been obtained. Moreover, variational principles for the HAE are formulated. The invariance identities of the HAE involving the Lagrangian L and the generators of the infinitesimal Lie group of transformations have been utilized for writing down their first integrals via Noether's theorem, Logan. We demonstrate the simplest example of the application of this technique, taking the box-shaped initial pulse and an ansatz based on linear Jost functions. We consider a combination of two boxes of opposite signs, the total area of the initial pulse being thus zero. Therewith, we develop a variational approximation for finding the eigenvalues of this pulse, by a piece-wise linear ansatz and tanh functions of the piece-wise linear function. Moreover, by using MATLAB, some graphical simulations were done to see the behavior of these solutions.

Keywords. Improved $\tan(\phi(\xi)/2)$-expansion method, Hamiltonian amplitude equation, Soliton wave solutions, Variational principles.
2010 Mathematics Subject Classification. 02.60.Lj, 02.70.Wz, 02.90.+p.

1. Introduction

In this paper, we consider the Hamiltonian amplitude equation as follows

$$iu_x + u_{tt} + 2\sigma|u|^2u - \varepsilon u_{xt} = 0, \tag{1.1}$$

where $\sigma = \pm 1$, $\varepsilon \ll 1$. The current equation was recently introduced by Wadati et al. [42]. This is an equation which governs certain instabilities of modulated wave trains, with the additional term u_{xt} overcoming the ill-posedness of the unstable nonlinear Schrödinger equation. It is a Hamiltonian analogue of the Kuramoto-Sivashinsky equation, which arises in dissipative systems and is apparently not integrable. In [15], the simplest equation method is used to construct the traveling wave solutions of the new Hamiltonian amplitude equation, $(3 + 1)$-dimensional generalized KP equation, Burgers-KP equation, coupled Higgs field equation, generalized Zakharov System. Demiray et al. [13], have applied the $(G'/G, 1/G)$-expansion method to obtain new exact traveling wave solutions of the Hamiltonian amplitude equations arise in the analysis of various problems in fluid mechanics, theoretical physics. Yan has obtained new

Received: 13 June 2024 ; Accepted: 12 July 2024.
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families of solitary wave solutions are found for a Hamiltonian amplitude equation by using a simple transformation and symbolic computation [43]. Mirzazadeh [35] obtained soliton solutions to the Hamiltonian amplitude equation by using the Heisenberg variational principle. Also, Chen et al. [6] has considered a new generalized Hamiltonian amplitude equation with nonlinear terms of any order by using a proper transformation and a generalized ansatz. In [20], coupled Higgs field equation and Hamiltonian amplitude equation are studied using the Lie classical method. The extended trial equation method and the generalized Kudryashov method are applied to find several exact solutions of the new Hamiltonian amplitude equation by Demiray and Bulut [12]. In fact, it has been discovered that many models in mathematics and physics are described by nonlinear partial differential equations. With the rapid development of nonlinear sciences based on computer algebraic systems, many effective methods have been presented. One of the most recent approaches is using semi-analytical methods [7–9, 38] or analytical methods [4, 5, 10, 11, 22–26, 29, 30] and machine learning methods [44], implications for aquifer systems [41], three-dimensional printing and digital rock physics [19], nonlinearities of SiGe bipolar phototransistor [14], study regarding the topological optimization [16], deep neural network based sentiment analysis [2], on Reinforced Concrete Bubble Deck Slabs [3]. So instead of using current models of partial differential equations, we can transfer PDEs to ordinary differential equations. Hence, there occurs a need to use a solitary wave variable that would appropriately transform PDEs to ODEs and solve them. In this paper, we apply the improved $\tan(\phi/2)$-expansion to solve the Hamiltonian amplitude equation. Many research papers dealing with analytical methods exist in the open literature, and some of them are reviewed and cited here for a better understanding of the present analysis [5, 7–11, 18, 24, 37, 38].
Authors of [34] explained the generalized fifth-order KdV-like equation with prime number $p = 3$ via a generalized bilinear differential operator. N-lump was investigated for the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [33]. Applications of $tan(\phi/2)$-expansion method for the Biswas-Milovic equation [29], the Gerdjikov-Ivanov model [28], the Kundu-Eckhaus equation [27], and the fifth-order integrable equations [21] were studied. Lump solutions were analyzed for the fractional generalized CBS-BK equation [45] and the (3+1)-D Burger system [17]. The approximations of a one-dimensional hyperbolic equation with non-local integral conditions were constructed by the reduced differential transform method [36]. The generalized Hirota bilinear strategy by the prime number was used for the (2+1)-dimensional generalized fifth-order KdV like equation [34]. The traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdV-mKdV equations were studied [1].
The paper is organized as follows: In section 2, we describe the improved $\tan(\phi(\xi)/2)$-expansion method. In section 3, we examine the new Hamiltonian amplitude equation with the method introduced in sections 2. Moreover, in section 4, we give the comparisons and numerical simulations of the solutions. Also, a conclusion is given in section 5.

2. Description of the ITEM

The ITEM is a well-known analytical method that was improved and developed by Manafian [29]. In this paper, we propose to develop this method, but prior to that, we give a detailed description of the method throughout the following steps:
Step 1. We suppose that the given nonlinear partial differential equation for u(x, t) is in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{2.1}$$

and can be converted to an ODE as:

$$\mathcal{Q}(u, ku', wu', k^2u'', w^2u'', ...) = 0, \tag{2.2}$$

by the transformation $\xi = kx + wt$ as the wave variable. Also, μ is a constant to be determined later.
Step 2. Suppose the traveling wave solution of Eq. (2.2) can be expressed as follows:

$$u(\xi) = \sum_{k=-m}^{m} A_k \left[p + \tan(\phi/2)\right]^k, \tag{2.3}$$

where $A_k(0 \le k \le m)$ and $A_{-k} = B_k(1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$ and $\phi = \phi(\xi)$ satisfies the following ordinary differential equation:

$$\phi'(\xi) = a\sin(\phi(\xi)) + b\cos(\phi(\xi)) + c. \tag{2.4}$$

We will consider the following special solutions of Equation (2.4):

Family 1: **When** $\Delta = a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]$.

Family 2: **When** $\Delta = a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right]$.

Family 3: **When** $\Delta = a^2+b^2-c^2 > 0$, $b \neq 0$ **and** $c = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b} + \frac{\sqrt{b^2+a^2}}{b}\tanh\left(\frac{\sqrt{b^2+a^2}}{2}\bar{\xi}\right)\right]$.

Family 4: **When** $\Delta = a^2+b^2-c^2 < 0$, $c \neq 0$ **and** $b = 0$, then $\phi(\xi) = 2\tan^{-1}\left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}\bar{\xi}\right)\right]$.

Family 5: **When** $\Delta = a^2+b^2-c^2 > 0$, $b-c \neq 0$ **and** $a = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right)\right]$.

Family 6: **When** $a = 0$ **and** c $= 0$, then $\phi(\xi) = \tan^{-1}\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}}+1}\right]$.

Family 7: **When** $b = 0$ **and** $c = 0$, then $\phi(\xi) = \tan^{-1}\left[\frac{2e^{a\bar{\xi}}}{e^{2a\bar{\xi}}+1}, \frac{e^{2a\bar{\xi}}-1}{e^{2a\bar{\xi}}+1}\right]$.

Family 8: **When** $a^2 + b^2 = c^2$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a\bar{\xi}+2}{(b-c)\bar{\xi}}\right]$.

Family 9: **When** $a = b = c = ka$, then $\phi(\xi) = 2\tan^{-1}\left[e^{ka\bar{\xi}} - 1\right]$.

Family 10: **When** $a = c = ka$ **and** b $= -$ka, then $\phi(\xi) = -2\tan^{-1}\left[\frac{e^{ka\bar{\xi}}}{-1+e^{ka\bar{\xi}}}\right]$.

Family 11: **When** $c = a$, then $\phi(\xi) = -2\tan^{-1}\left[\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right]$.

Family 12: **When** $a = c$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right]$.

Family 13: **When** $c = -a$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right]$.

Family 14: **When** $b = -c$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{ae^{a\bar{\xi}}}{1-ce^{a\bar{\xi}}}\right]$.

Family 15: **When** $b = 0$ **and** a $=$ c, then $\phi(\xi) = -2\tan^{-1}\left[\frac{c\bar{\xi}+2}{c\bar{\xi}}\right]$.

Family 16: **When** $a = 0$ **and** $b = c$, then $\phi(\xi) = 2\tan^{-1}\left[c\bar{\xi}\right]$.

Family 17: **When** $a = 0$ **and** $b = -c$, then $\phi(\xi) = -2\tan^{-1}\left[\frac{1}{c\bar{\xi}}\right]$.

Family 18: **When** $a = 0$ **and** $b = 0$, then $\phi(\xi) = c\xi + C$.

Family 19: **When** $b = c$ then $\phi(\xi) = 2\tan^{-1}\left[\frac{e^{a\bar{\xi}}-c}{a}\right]$,

where $\bar{\xi} = \xi + C, p, A_0, A_k, B_k (k = 1, 2, ..., m), a, b$ and c are constants to be determined later.

Step 3. To determine m. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest-order nonlinear term(s) in Eq. (2.2). But, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.2). If $m = q/p$ (where $m = q/p$ is a fraction in the lowest terms), we let

$$u(\xi) = v^{q/p}(\xi), \tag{2.5}$$

then substitute Eq. (2.5) into Eq. (2.2) and then determine the value of m in new Eq. (2.2). If m is a negative integer, we let

$$u(\xi) = v^m(\xi), \tag{2.6}$$

then substitute Eq. (2.6) into Eq. (2.2). Then we determine the new value of m in the obtained equation. Moreover, precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = m$, which gives rise to the degree of another expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = m + q, \qquad D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = mp + s(m + q). \tag{2.7}$$

Step 4. Substituting (2.3) into Eq. (2.2) with the value of m obtained in Step 2. Collecting the coefficients of $\tan(\phi/2)^k$, $\cot(\phi/2)^k (k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined equations

for $A_0, A_k, B_k (k = 1, 2, ..., m)$ a, b, c and p with the aid of symbolic computation Maple.
Step 5. Solving the algebraic equations in Step 3, then substituting
$A_0, A_1, B_1, ..., A_m, B_m, \mu, p$ in (2.3).

3. The Hamiltonian amplitude equation

We consider the Hamiltonian amplitude equation as follows

$$iu_x + u_{tt} + 2\sigma|u|^2u - \varepsilon u_{xt} = 0, \tag{3.1}$$

where $\sigma = \pm 1$, $\varepsilon \ll 1$. By make the transformation

$$u(x,t) = e^{i\eta}v(\xi), \qquad \eta = \alpha x + \beta t, \qquad \xi = \mu(x - st), \tag{3.2}$$

the Eq. (3.2) is carried to an ODE

$$(\mu^2 s^2 + \varepsilon\mu^2 s)v'' + i(\mu - 2\beta\mu s - \varepsilon\beta\mu + \varepsilon\alpha\mu s)v' - (\alpha + \beta^2 - \varepsilon\alpha\beta)v + 2\sigma v^3 = 0. \tag{3.3}$$

If we take

$$s = \frac{1 - \varepsilon\beta}{2\beta - \alpha\varepsilon}, \tag{3.4}$$

then Eq. (3.3) transform into

$$(\mu^2 s^2 + \varepsilon\mu^2 s)v'' - (\alpha + \beta^2 - \varepsilon\alpha\beta)v + 2\sigma v^3 = 0. \tag{3.5}$$

Also, we know

$$v(\xi) = A_m \left(\tan(\phi(\xi)/2)\right)^m + ..., \tag{3.6}$$

$$\frac{dv(\xi)}{d\xi} = \frac{m(c-b)}{2} A_m \left(\tan(\phi(\xi)/2)\right)^{m+1} + ..., \tag{3.7}$$

$$\frac{d^2v(\xi)}{d\xi^2} = \frac{m(m+1)(c-b)^2}{2} A_m \left(\tan(\phi(\xi)/2)\right)^{m+2} + ..., \tag{3.8}$$

$$v^3(\xi) = A_m^3 \left(\tan(\phi(\xi)/2)\right)^{3m} + ... \tag{3.9}$$

Balancing the v'' and v^3, using homogenous principle, we get

$$m + 2 = 3m, \qquad \Rightarrow m = 1. \tag{3.10}$$

The close form of solution is

$$v(\xi) = A_0 + A_1\left[p + \tan(\phi/2)\right] + B_1\left[p + \tan(\phi/2)\right]^{-1}. \tag{3.11}$$

Substituting (3.11) and (2.4) into Eq. (3.5) and by using the well-known Maple software, we have the following sets of coefficients for the solutions of (3.11) as coefficients of $Y = \tan(\phi/2)$:

$$\begin{aligned}
Y^0 \quad &: \quad 2p^3d_0(\epsilon\alpha\beta - \alpha - \beta^2) + 4\sigma(e_1^3 + 3p^2e_1^2d_1 + 3p^4e_1d_1^2 + p^6d_1^3 + 3d_0pe_1^2 \\
&+6d_0p^3e_1d_1 + 3d_0p^5d_1^2 + d_0^3p^3) + p^3d_1(2p\epsilon\alpha\beta + 12pd_0^2\sigma - 2p\alpha - 2p\beta^2 + \mu^2s^2ab \\
&+\mu^2s\epsilon ac + \mu^s\epsilon ab + \mu^2s^2ac) - e_1(2\beta^2p^2 + \mu^2s^2abp - 2\epsilon\alpha\beta p^2\mu^2s\epsilon abp - 12\sigma d_0^2p^2 \\
&+\mu^2(s^2acp + s\epsilon acp - s\epsilon b^2 - s\epsilon c^2 - s^2b^2 - 2s^2bc - 2s\epsilon bc - s^2c^2) + 2\alpha p^2) = 0, \\
Y^1 \quad &: \quad 12p^4\sigma d_1^2(2pd_1 + 5d_0) + p^3d_1(48\sigma e_1d_1 + 2\mu^2s\epsilon a^2 - 8\alpha + 4\sigma d_0^2 - 8\beta^2 \\
&+\mu^2(2s^2a^2 + s\epsilon c^2 - s^2b^2 + s^2c^2 - s\epsilon b^2) + 8\epsilon\alpha\beta) + 3p^2(\mu^2s^2acd_1 + 24\sigma d_0e_1d_1 \\
&+\mu^2s^2abd_1 + \mu^2s\epsilon acd_1 - 2\beta^2d_0 + \mu^2s\epsilon abd_1 + 2\epsilon\alpha\beta d_0 + 4\sigma d_0^3 - 2\alpha d_0) \\
&-pe_1(-24\sigma e_1d_1 - 4\epsilon\alpha\beta + \mu^2s^2c^2 - \mu^2s^2b^2 + 2\mu^2s^2a^2 + 4\alpha - \mu^2s\epsilon b^2 + 4\beta^2 + \mu^2s\epsilon c^2 \\
&+2\mu^2s\epsilon a^2 - 24\sigma d_0^2)3e_1(4\sigma d_0e_1 + \mu^2s^2ab + \mu^2s\epsilon ab + \mu^2s^2ac + \mu^2s\epsilon ac) = 0,
\end{aligned}$$

$$
\begin{aligned}
Y^2 \quad &: \quad 60\sigma d_1^3 p^4 - 3d_1 p^3(-40d_0\sigma d_1 + \mu^2 s\epsilon ab - \mu^2 s^2 ac + \mu^2 s^2 ab - \mu^2 s\epsilon ac) \\
&\quad +3p^2 d_1(24\sigma e_1 d_1 + c^2 s^2\mu^2 + c^2\epsilon s\mu^2 - 4\alpha - b^2 s^2\mu^2 - 4\beta^2 + 2a^2 s^2\mu^2 + 4\epsilon\alpha\beta \\
&\quad -b^2\epsilon s\mu^2 + 24\sigma d_0^2 + 2a^2\epsilon s\mu^2) + 3p(-2\alpha d_0 + \mu^2 s^2 d_1 ab - \mu^2 s\epsilon e_1 ac + 4\sigma d_0^3 \\
&\quad +2\epsilon\alpha\beta d_0 + \mu^2 s^2 d_1 ac + \mu^2 s\epsilon e_1 ab + 24\sigma d_0 d_1 e_1 - 2\beta^2 d_0 + \mu^2 s\epsilon d_1 ac - \mu^2 s^2 e_1 ac \\
&\quad +\mu^2 s^2 e_1 ab + \mu^2 s\epsilon d_1 ab) + e_1(12\sigma e_1 d_1 - b^2 s^2\mu^2 - 2\alpha + 2\epsilon\alpha\beta + c^2\epsilon s\mu^2 \\
&\quad -2\beta^2 - b^2\epsilon s\mu^2 + 2a^2\epsilon s\mu^2 + 2a^2 s^2\mu^2 + c^2 s^2\mu^2 + 12\sigma d_0^2) = 0, \qquad (3.12)\\
Y^3 \quad &: \quad p^3 d_1(80\sigma d_1^2 + b^2 s^2\mu^2 + c^2 s^2\mu^2 - 2cbs^2\mu^2 + b^2\epsilon s\mu^2 + c^2\epsilon s\mu^2 - 2bc\epsilon s\mu^2) \\
&\quad -3d_1 p^2(-40d_0\sigma d_1 + 3\mu^2 s\epsilon ab - 3\mu^2 s\epsilon ac - 3\mu^2 s^2 ac + 3\mu^2 s^2 ab) + p(48\sigma d_1^2 e_1 + 48\sigma d_0^2 d_1 \\
&\quad -3\mu^2 s\epsilon b^2 d_1 + 3\mu^2 s\epsilon c^2 d_1 + 6\mu^2 s^2 a^2 d_1 - 8\beta^2 d_1 + 6\mu^2 s\epsilon a^2 d_1 - 3\mu^2 s^2 b^2 d_1 \\
&\quad +8\epsilon\alpha\beta d_1 - 8\alpha d_1 + 3\mu^2 s^2 c^2 d_1 + 2\mu^2 s^2 bce_1 - \mu^2 s^2 b^2 e_1 - \mu^2 s^2 c^2 e_1 - \mu^2 s\epsilon b^2 e_1 - \mu^2 s\epsilon c^2 e_1 \\
&\quad +2\mu^2 s\epsilon bce_1) - 2\alpha d_0 - \mu^2 s\epsilon e_1 ab + \mu^2 s^2 d_1 ac + \mu^2 s^2 d_1 ab + \mu^2 s^2 e_1 ac \\
&\quad +4\sigma d_0^3 - 2\beta^2 d_0 + \mu^2 s\epsilon d_1 ac + \mu^2 s\epsilon e_1 ac + 2\epsilon\alpha\beta d_0 + 24\sigma d_0 d_1 e_1 + \mu^2 s\epsilon d_1 ab - \mu^2 s^2 e_1 ab = 0, \\
Y^4 \quad &: \quad 3p^2 d_1(20\sigma d_1^2 + \mu^2 s\epsilon b^2 + \mu^2 s\epsilon c^2 + \mu^2 s^2 b^2 - 2cb\epsilon s\mu^2 + \mu^2 s^2 c^2 - 2cbs^2\mu^2) \\
&\quad -3d_1 p(-20d_0\sigma d_1 - 3\mu^2 s^2 ac + 3\mu^2 s^2 ab - 3\mu^2 s\epsilon ac + 3\mu^2 s\epsilon ab) \\
&\quad +d_1(12\sigma e_1 d_1 - 2\beta^2 + 12\sigma d_0^2 - \mu^2 s^2 b^2 - \mu^2 s\epsilon b^2 + 2\epsilon\alpha\beta + 2\mu^2 s^2 a^2 \\
&\quad +2\mu^2 s\epsilon a^2 + \mu^2 s^2 c^2 + \mu^2 s\epsilon c^2 - 2\alpha) = 0, \\
Y^5 \quad &: \quad 3pd_1(8\sigma d_1^2 + \mu^2 s\epsilon b^2 + \mu^2 s\epsilon c^2 + \mu^2 s^2 b^2 - 2cb\epsilon s\mu^2 + \mu^2 s^2 c^2 - 2cbs^2\mu^2) \\
&\quad -3d_1(-4d_0\sigma d_1 - \mu^2 s\epsilon ac + \mu^2 s\epsilon ab + \mu^2 s^2 ab - \mu^2 s^2 ac) = 0, \\
Y^6 \quad &: \quad d_1(4\sigma d_1^2 + \mu^2 s\epsilon b^2 + \mu^2 s\epsilon c^2 + \mu^2 s^2 b^2 - 2cb\epsilon s\mu^2 + \mu^2 s^2 c^2 - 2cbs^2\mu^2) = 0.
\end{aligned}
$$

Solving the algebraic equations using Maple, we get the following results:
Case I:

$$s = s, \quad A_0 = 0, \quad A_1 = 0, \quad B_1 = B_1, \quad \mu = \pm\frac{2(b-c)}{\Delta}\sqrt{\frac{-\sigma}{s^2 + s\varepsilon}}B_1, \quad p = -\frac{a}{b-c}, \tag{3.13}$$

$$\Delta = a^2 + b^2 - c^2, \quad \beta = \beta, \quad \alpha = \frac{\Delta\beta^2 + 2\sigma(2bc - b^2 - c^2)B_1^2}{\Delta(\varepsilon\beta - 1)}, \quad u(\xi) = B_1\left[-\frac{a}{b-c} + \tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1} e^{i\eta}. \tag{3.14}$$

By using the (3.14) and **Families 1, 2, 6, 8, 10-15** and **17**, respectively, we get

$$u_1(\xi) = -\frac{B_1(b-c)}{\sqrt{-\Delta}}\cot\left[\frac{\sqrt{-\Delta}}{2}(\xi + C)\right]e^{i\eta}, \qquad u_2(\xi) = \frac{(b-c)B_1}{\sqrt{\Delta}}\coth\left[\frac{\sqrt{\Delta}}{2}(\xi + C)\right]e^{i\eta}, \tag{3.15}$$

$$u_3(\xi) = B_1\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}\right]\right)e^{i\eta}, \quad u_4(\xi) = \frac{B_1(b-c)(\xi + C)}{a(\xi + C) + 2}e^{i\eta}, \tag{3.16}$$

$$u_5(\xi) = B_1\left[\frac{1}{2} - \frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)} - 1\right]}\right]^{-1} e^{i\eta}, \qquad u_6(\xi) = B_1\left[\frac{a}{a-b} - \frac{(a+b)e^{b(\xi+C)} - 1}{(a-b)e^{b(\xi+C)} - 1}\right]^{-1} e^{i\eta}, \tag{3.17}$$

$$u_7(\xi) = B_1\left[\frac{c}{c-b} + \frac{(b+c)e^{b(\xi+C)} + 1}{(b-c)e^{b(\xi+C)} - 1}\right]^{-1} e^{i\eta}, \qquad u_8(\xi) = B_1\left[-\frac{a}{a+b} + \frac{e^{b(\xi+C)} + b - a}{e^{b(\xi+C)} - b - a}\right]^{-1} e^{i\eta},$$

$$u_9(\xi) = B_1\left[\frac{a}{2c} - \frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)} - 1}\right]^{-1} e^{i\eta}, \qquad u_{10}(\xi) = B_1\left[1 - \frac{c(\xi + C) + 2}{c(\xi + C)}\right]^{-1} e^{i\eta}, \tag{3.18}$$

$$u_{11}(\xi) = -cB_1(\xi + C)e^{i\eta},$$

where $\xi=\pm\frac{2B_1(b-c)}{a^2+b^2-c^2}\sqrt{\frac{-\sigma}{s^2+s\varepsilon}}(x-st)$, $\eta=\left(\frac{(a^2+b^2-c^2)\beta^2+2\sigma(2bc-b^2-c^2)B_1^2}{(a^2+b^2-c^2)(\varepsilon\beta-1)}\right)x+\beta t$, and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case II:

$$A_0=0,\qquad A_1=0,\qquad B_1=\frac{1}{(b-c)\varepsilon}\sqrt{\frac{\Delta}{2\sigma}},\qquad \mu=\pm\frac{1}{\varepsilon}\sqrt{\frac{-2}{\Delta(s^2+s\varepsilon)}},\tag{3.19}$$

$$s=s,\quad p=-\frac{a}{b-c},\quad \beta=\frac{1}{\varepsilon},\quad \alpha=\alpha,\quad u(\xi)=\frac{1}{(b-c)\varepsilon}\sqrt{\frac{\Delta}{2\sigma}}\left[-\frac{a}{b-c}+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}e^{i\eta}.\tag{3.20}$$

By using the (3.20) and **Families 1, 2, 6** and **10-14**, respectively, we can write

$$u_{12}(\xi)=-\frac{1}{\varepsilon\sqrt{-2\sigma}}\cot\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)e^{i\eta},\qquad u_{13}(\xi)=\frac{1}{\varepsilon\sqrt{2\sigma}}\coth\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)e^{i\eta},\tag{3.21}$$

$$u_{14}(\xi)=\frac{1}{\varepsilon\sqrt{2\sigma}}\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)e^{i\eta},\tag{3.22}$$

$$u_{15}(\xi)=-\frac{1}{2\varepsilon\sqrt{2\sigma}}\left[\frac{1}{2}+\frac{e^{ka(\xi+C)}}{1-e^{ka(\xi+C)}}\right]^{-1}e^{i\eta},$$

$$u_{16}(\xi)=\frac{b}{(b-a)\varepsilon\sqrt{2\sigma}}\left[\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1}e^{i\eta},\tag{3.23}$$

$$u_{17}(\xi)=\frac{b}{(b-c)\varepsilon\sqrt{2\sigma}}\left[\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}e^{i\eta},$$

$$u_{18}(\xi)=\frac{b}{(b+a)\varepsilon\sqrt{2\sigma}}\left[-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1}e^{i\eta},$$

$$u_{19}(\xi)=-\frac{a}{2c\varepsilon\sqrt{2\sigma}}\left[\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}e^{i\eta}.$$

where $\xi=\pm\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+b^2-c^2)(s^2+s\varepsilon)}}(x-st)$, $\eta=\alpha x+\frac{1}{\varepsilon}t$, and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case III:

$$A_0=\pm\mu(a+(b-c)p)\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}},\qquad A_1=0,\qquad B_1=\mp\mu(2ap+(b-c)p^2-b-c)\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}},\tag{3.24}$$

$$\Delta=a^2+b^2-c^2,\quad \mu=\mu,\quad s=s,\quad p=p,\quad \beta=\beta,\quad \alpha=\frac{\Delta(s^2\mu^2+\varepsilon s\mu^2)+\beta^2}{2(\varepsilon\beta-1)},\tag{3.25}$$

$$u(\xi)=\pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(b-c)p]-[2ap+(b-c)p^2-b-c]\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}.$$

By using the (3.25) and **Families 1, 2, 6-15** and **17**, respectively, one should be write

$$u_{20}(\xi)=\pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\{a+(b-c)p-[2ap+(b-c)p^2-b-c]\times[p+\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan(\frac{\sqrt{-\Delta}}{2}(\xi+C))]^{-1}\}e^{i\eta},\tag{3.26}$$

$$u_{21}(\xi)=\pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\{[a+(b-c)p]-[2ap+(b-c)p^2-b-c]\times[p+\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh(\frac{\sqrt{\Delta}}{2}(\xi+C))]^{-1}\}e^{i\eta},\tag{3.27}$$

$$u_{22}(\xi) = \pm\mu b\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{p-(p^2-1)\left[p+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\ \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]^{-1}\right\}e^{i\eta}, \tag{3.28}$$

$$u_{23}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{1-2p\left[p+\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1},\ \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]\right)\right]^{-1}\right\}e^{i\eta},$$

$$u_{24}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(b-c)p]-[2ap+(b-c)p^2-b-c]\left[p+\frac{a(\xi+C)+2}{(b-c)(\xi+C)}\right]^{-1}\right\}e^{i\eta},$$

$$u_{25}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}ka\left\{1-2(p-1)\left[p+\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^{-1}\right\}e^{i\eta}, \tag{3.29}$$

$$u_{26}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}ka\left\{1-2p-2p(1-p)\left[p-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{27}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(b-a)p]-[2ap+(b-a)p^2-b-a]\left[p-\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{28}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[c+(b-c)p]-[2cp+(b-c)p^2-b-c]\left[p+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]^{-1}\right\}e^{i\eta},$$

$$u_{29}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a+(b+a)p]-[2ap+(b+a)p^2-b+a]\left[p+\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{30}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[a-2cp]-[2ap-2cp^2]\left[p-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{31}(\xi) = \pm\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{[c-cp]-[2cp-cp^2-c]\left[p-\frac{c(\xi+C)+2}{c(\xi+C)}\right]^{-1}\right\}e^{i\eta}, \tag{3.30}$$

$$u_{32}(\xi) = \mp 2c\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\,[p+c(\xi+C)]^{-1}\,e^{i\eta},$$

$$u_{33}(\xi) = \mp 2cp\mu\sqrt{-\frac{s^2+s\varepsilon}{4\sigma}}\left\{1+p\left[p-\frac{1}{c(\xi+C)}\right]^{-1}\right\}e^{i\eta},$$

where $\xi=\mu(x-st)$, $\eta=\left(\frac{(a^2+b^2-c^2)(s^2\mu^2+\varepsilon s\mu^2)+\beta^2}{2(\varepsilon\beta-1)}\right)x+\beta t$, and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case IV:

$$\beta=\beta,\quad s=s,\quad p=0,\quad \mu=\frac{2A_1}{b-c}\sqrt{-\frac{\sigma}{s^2+\varepsilon s}},\qquad A_0=-\frac{aA_1}{b-c},\qquad A_1=A_1,\qquad B_1=0, \tag{3.31}$$

$$\Delta=\mathrm{a}^2+\mathrm{b}^2-\mathrm{c}^2,\quad \alpha=-\frac{2\Delta\sigma\mathrm{A}_1^2-\beta^2(\mathrm{b}-\mathrm{c})^2}{(\mathrm{b}-\mathrm{c})^2(\varepsilon\beta-1)},\quad \mathrm{u}(\xi)=\mathrm{A}_1\left[-\frac{\mathrm{a}}{\mathrm{b}-\mathrm{c}}+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]\mathrm{e}^{\mathrm{i}\eta}. \tag{3.32}$$

By using the (3.32) and **Families 1, 2, 6, 8, 10-14,** and **17**, respectively, we can get

$$u_{34}(\xi)=-A_1\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)e^{i\eta},\qquad u_{35}(\xi)=A_1\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)e^{i\eta}, \tag{3.33}$$

$$u_{36}(\xi) = A_1 \tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)e^{i\eta}, \tag{3.34}$$

$$u_{37}(\xi) = A_1\left[-\frac{a}{b-c}+\frac{a(\xi+C)+2}{(b-c)(\xi+C)}\right]e^{i\eta}.$$

$$u_{38}(\xi) = A_1\left[\frac{1}{2}-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]e^{i\eta}, \quad u_{39}(\xi) = A_1\left[\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]e^{i\eta}, \tag{3.35}$$

$$u_{40}(\xi) = A_1\left[\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]e^{i\eta}, \quad u_{41}(\xi) = A_1\left[-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right]e^{i\eta},$$

$$u_{42}(\xi) = A_1\left[\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]e^{i\eta}, \quad u_{43}(\xi) = A_1\left[1-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right]e^{i\eta}, \quad u_{44}(\xi) = \frac{-A_1e^{i\eta}}{c(\xi+C)},$$

where $\xi = \frac{2A_1}{b-c}\sqrt{-\frac{\sigma}{s^2+\varepsilon s}}(x-st)$, $\eta = \left(-\frac{2(a^2+b^2-c^2)\sigma A_1^2-\beta^2(b-c)^2}{(b-c)^2(\varepsilon\beta-1)}\right)x+\beta t$, and $s = \frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case V:

$$\Delta = a^2+b^2-c^2, \quad p = \frac{1}{b-c}\left(\sqrt{\frac{\Delta}{3}}-a\right), \quad \mu = \frac{1}{\varepsilon}\sqrt{\frac{-2}{\Delta(s^2+\varepsilon s)}}, \quad A_0 = \pm\frac{1}{\varepsilon\sqrt{6\sigma}}, \quad s = s, \quad \beta = \frac{1}{\varepsilon}, \tag{3.36}$$

$$A_1 = 0, \quad B_1 = \pm\frac{1}{(b-c)\varepsilon}\sqrt{\frac{2\Delta}{9\sigma}}, \quad \alpha = \alpha, \tag{3.37}$$

$$u(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{(b-c)\varepsilon}\sqrt{\frac{2\Delta}{9\sigma}}\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}.$$

By using the (3.37) and **Families 1, 2, 6,** and **10-14**, respectively, give as

$$u_{45}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\sqrt{\frac{1}{3}}-\sqrt{-1}\tan\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta}, \tag{3.38}$$

$$u_{46}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{1}{\sqrt{3}}+\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)\right]^{-1}\right\}e^{i\eta}, \tag{3.39}$$

$$u_{47}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{1}{\sqrt{3}}+\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}, \frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right]^{-1}\right\}e^{i\eta}, \tag{3.40}$$

$$u_{48}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{1}{\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{1}{\sqrt{3}}-1-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right]^{-1}\right\}e^{i\eta}, \tag{3.41}$$

$$u_{49}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{b}{(b-a)\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{b}{(b-a)\sqrt{3}}-\frac{a}{b-a}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{50}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{b}{(b-c)\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{b}{(b-c)\sqrt{3}}-\frac{c}{b-c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{51}(\xi) = \left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{b}{(b+a)\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{b}{(b+a)\sqrt{3}}-\frac{a}{b+a}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

$$u_{52}(\xi)=\left\{\pm\frac{1}{\varepsilon\sqrt{6\sigma}}\pm\frac{a}{2c\varepsilon}\sqrt{\frac{2}{9\sigma}}\left[\frac{a}{2c\sqrt{3}}-\frac{a}{2c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1}\right\}e^{i\eta},$$

where $\xi=\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+b^2-c^2)(s^2+\varepsilon s)}}(x-st)$, $\eta=\alpha x+\frac{1}{\varepsilon}t$, and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case VI:

$$\Delta=a^2+b^2-c^2,\quad p=0,\quad \mu=\frac{1}{\varepsilon}\sqrt{\frac{-2}{\Delta(s^2+\varepsilon s)}},\qquad A_0=\mp\frac{a}{\varepsilon}\sqrt{\frac{2}{\sigma\Delta}},\qquad s=s,\quad \beta=\frac{1}{\varepsilon},\tag{3.42}$$

$$\mathrm{A}_1=\pm\frac{\mathrm{b}-\mathrm{c}}{\varepsilon\sqrt{2\sigma\Delta}},\qquad \mathrm{B}_1=0,\qquad \alpha=\alpha,\qquad \mathrm{u}(\xi)=\left\{\mathrm{A}_0+\mathrm{A}_1\tan\left(\frac{\Phi(\xi)}{2}\right)\right\}\mathrm{e}^{\mathrm{i}\eta}.\tag{3.43}$$

By using the (3.43) and **Families 1, 2, 6,** and **10-14**, respectively, we can get

$$u_{53}(\xi)=\mp\frac{1}{\varepsilon}\frac{1}{\sqrt{\sigma\Delta}}\left\{a\sqrt{2}-\frac{1}{\sqrt{2}}\left[a-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)\right]\right\}e^{i\eta},\tag{3.44}$$

$$u_{54}(\xi)=\mp\frac{1}{\varepsilon}\frac{1}{\sqrt{\sigma\Delta}}\left\{a\sqrt{2}-\frac{1}{\sqrt{2}}\left[a+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)\right]\right\}e^{i\eta},\tag{3.45}$$

$$u_{55}(\xi)=\pm\frac{1}{\varepsilon\sqrt{2\sigma}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)e^{i\eta}.\tag{3.46}$$

$$\begin{aligned}
u_{56}(\xi)&=\pm\frac{1}{\varepsilon}\sqrt{\frac{2}{\sigma}}\left\{1-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right\}e^{i\eta},\\
u_{57}(\xi)&=\pm\frac{1}{b\varepsilon}\sqrt{\frac{2}{\sigma}}\left\{2a-(b-a)-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right\}e^{i\eta},\\
u_{58}(\xi)&=\pm\frac{1}{b\varepsilon}\sqrt{\frac{2}{\sigma}}\left\{2c-(b-c)+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right\}e^{i\eta},\\
u_{59}(\xi)&=\pm\frac{1}{b\varepsilon}\sqrt{\frac{2}{\sigma}}\left\{2a-(b+a)+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right\}e^{i\eta},\\
u_{60}(\xi)&=\pm\frac{2}{\varepsilon}\sqrt{\frac{2}{\sigma}}\left\{1+\frac{c}{a}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right\}e^{i\eta},
\end{aligned}\tag{3.47}$$

where $\xi=\frac{1}{\varepsilon}\sqrt{\frac{-2}{(a^2+b^2-c^2)(s^2+\varepsilon s)}}(x-st)$, $\eta=\alpha x+\frac{1}{\varepsilon}t$, and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case VII:

$$\Delta=a^2+b^2-c^2,\quad s=s,\quad \beta=\beta,\quad p=-\frac{a}{b-c},\quad \mu=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{\Delta(s^2+\varepsilon s)}},\tag{3.48}$$

$$A_0=0,\quad A_1=(b-c)\sqrt{\frac{\varepsilon\alpha\beta-\alpha-\beta^2}{4\sigma\Delta}},$$

$$B_1=-\frac{\sqrt{\Delta(\varepsilon\alpha\beta-\alpha-\beta^2)}}{\sqrt{4\sigma}(b-c)},\quad \alpha=\alpha,\quad u(\xi)=\left\{A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}.\tag{3.49}$$

By using the (3.49) and **Families 1, 2, 6,** and **10-14**, respectively, one should be write

$$u_{61}(\xi)=-\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\tan\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)\right\}e^{i\eta},\tag{3.50}$$

$$u_{62}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{3.51}$$

$$u_{63}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right.$$
$$\left.-\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right\}e^{i\eta}, \tag{3.52}$$

$$u_{64}(\xi)=-\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{2\left(\frac{1}{2}-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)^{-1}\right\}e^{i\eta}, \tag{3.53}$$

$$u_{65}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\frac{b-a}{b}\left(\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right)-\frac{b}{b-a}\left(\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{66}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\frac{b-c}{b}\left(\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right)-\frac{b}{b-c}\left(\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{67}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\frac{b+a}{b}\left(-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right)-\frac{b}{b+a}\left(-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right)^{-1}\right\}e^{i\eta},$$

$$u_{68}(\xi)=-\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left\{\frac{2c}{a}\left(\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)-\frac{a}{2c}\left(\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

where $\xi=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{(a^2+b^2-c^2)(s^2+\varepsilon s)}}(x-st)$, $\eta=\alpha x+\beta t$, and $s=\frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case VIII:

$$\Delta=a^2+b^2-c^2,\quad s=s,\quad \alpha=\alpha,\quad \beta=\beta,\quad p=-\frac{a}{b-c},\quad \mu=\sqrt{\frac{2(\varepsilon\alpha\beta-\alpha-\beta^2)}{\Delta(s^2+\varepsilon s)}},\quad A_0=0, \tag{3.54}$$

$$A_1=(b-c)\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma\Delta}},\quad B_1=0,\quad u(\xi)=A_1\left[-\frac{a}{b-c}+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]e^{i\eta}, \tag{3.55}$$

By using the (3.55) and **Families 1, 2, 6,** and **10-14** can be written respectively as

$$u_{69}(\xi)=-\sqrt{-\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\tan\left(\frac{\sqrt{-\Delta}}{2}(\xi+C)\right)e^{i\eta},\qquad u_{70}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+C)\right)e^{i\eta}, \tag{3.56}$$

$$u_{71}(\xi)=\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)e^{i\eta}, \tag{3.57}$$

$$u_{72}(\xi)=-\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{\sigma}}\left(\frac{1}{2}-\frac{e^{ka(\xi+C)}}{\left[e^{ka(\xi+C)}-1\right]}\right)e^{i\eta}, \tag{3.58}$$

$$u_{73}(\xi)=\frac{b-a}{b}\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left(\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right)e^{i\eta},$$

$$u_{74}(\xi)=\frac{b-c}{b}\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left(\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right)e^{i\eta},$$

$$u_{75}(\xi) = \frac{b+a}{b}\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left(-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right)e^{i\eta},$$

$$u_{76}(\xi) = -\frac{2c}{a}\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{4\sigma}}\left(\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)e^{i\eta},$$

where $\xi = \sqrt{\frac{2(\varepsilon\alpha\beta-\alpha-\beta^2)}{(a^2+b^2-c^2)(s^2+\varepsilon s)}}(x-st)$, $\eta = \alpha x+\beta t$ and $s = \frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Case IX:

$$\Delta = a^2+b^2-c^2, \quad p = -\frac{a}{b-c}, \quad \mu = \sqrt{\frac{\varepsilon\alpha\beta-\alpha-\beta^2}{2\Delta(s^2+\varepsilon s)}}, \quad s=s, \quad A_0=0, \quad A_1 = \pm(b-c)\sqrt{\frac{\varepsilon\alpha\beta-\alpha-\beta^2}{8\sigma\Delta}}, \tag{3.59}$$

$$B_1 = \mp\frac{1}{b-c}\sqrt{\frac{(\varepsilon\alpha\beta-\alpha-\beta^2)\Delta}{8\sigma}}, \quad \beta=\beta, \quad \alpha=\alpha, \quad u(\xi) = \left\{A_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]+B_1\left[p+\tan\left(\frac{\Phi(\xi)}{2}\right)\right]^{-1}\right\}e^{i\eta}. \tag{3.60}$$

By using the (3.60) and **Families 1, 2, 6,** and **10-14**, respectively, get as

$$u_{77}(\xi) = \mp\sqrt{-\frac{\varepsilon\alpha\beta-\alpha-\beta^2}{8\sigma}}\left\{\tan\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)-\cot\left(\frac{\sqrt{c^2-b^2-a^2}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{3.61}$$

$$u_{78}(\xi) = \pm\sqrt{\frac{\varepsilon\alpha\beta-\alpha-\beta^2}{8\sigma}}\left\{\tanh\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)-\coth\left(\frac{\sqrt{a^2+b^2-c^2}}{2}(\xi+C)\right)\right\}e^{i\eta}, \tag{3.62}$$

$$\begin{aligned} u_{79}(\xi) = \pm\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{8\sigma}}&\left\{\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right. \\ &\left.-\cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1},\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}\right]\right)\right\}e^{i\eta}, \end{aligned} \tag{3.63}$$

$$u_{80}(\xi) = \mp\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{8\sigma}}\left\{2\left(\frac{1}{2}-\frac{e^{ka(\xi+C)}}{[e^{ka(\xi+C)}-1]}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{e^{ka(\xi+C)}}{[e^{ka(\xi+C)}-1]}\right)^{-1}\right\}e^{i\eta}, \tag{3.64}$$

$$u_{81}(\xi) = \pm\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{8\sigma}}\left\{\frac{b-a}{b}\left(\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right)-\frac{b}{b-a}\left(\frac{a}{a-b}-\frac{(a+b)e^{b(\xi+C)}-1}{(a-b)e^{b(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{82}(\xi) = \pm\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{8\sigma}}\left\{\frac{b-c}{b}\left(\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right)-\frac{b}{b-c}\left(\frac{c}{c-b}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

$$u_{83}(\xi) = \pm\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{8\sigma}}\left\{\frac{b+a}{b}\left(-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right)-\frac{b}{b+a}\left(-\frac{a}{a+b}+\frac{e^{b(\xi+C)}+b-a}{e^{b(\xi+C)}-b-a}\right)^{-1}\right\}e^{i\eta},$$

$$u_{84}(\xi) = \mp\sqrt{\frac{\alpha+\beta^2-\varepsilon\alpha\beta}{8\sigma}}\left\{\frac{2c}{a}\left(\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)-\frac{a}{2c}\left(\frac{a}{2c}-\frac{ae^{a(\xi+C)}}{ce^{a(\xi+C)}-1}\right)^{-1}\right\}e^{i\eta},$$

where $\xi = \sqrt{\frac{\varepsilon\alpha\beta-\alpha-\beta^2}{2(a^2+b^2-c^2)(s^2+\varepsilon s)}}(x-st)$, $\eta = \alpha x+\beta t$, and $s = \frac{1-\varepsilon\beta}{2\beta-\alpha\varepsilon}$.

Remark 3.1. Eslami and Mirzazadeh [15] studied the exact solutions of the Hamiltonian amplitude equation through the simplest equation method and found only three soliton solutions as singular exponential wave solution ((20) in [15]), soliton wave solution ((23) in [15]). Similarly, it can be shown that Demiray et al. [13] with $(G'/G, 1/G)$-expansion method has obtained some solutions, including travelling wave solution ((3.1.9) and (3.1.14) in [13]), solitary wave solution ((3.1.11) and (3.1.12) in [13]), periodic wave solution ((3.1.16) and (3.1.17) in [13]) and rational wave solution ((3.1.19) in [13]). On the other hand, by means of the ITEM, we have obtained 84 solutions for the Hamiltonian amplitude equation. Our solutions with ITEM are include hyperbolic, periodic, singular kink and rational solutions. Moreover, for particular values of the free parameters, some of our solutions coincide with solutions of Wazwaz [15]. It proves that the other solutions are newly derived through the improved $\tan(\phi(\xi)/2)$-expansion method. Moreover, the numerical simulations of the Hamiltonian amplitude equation will

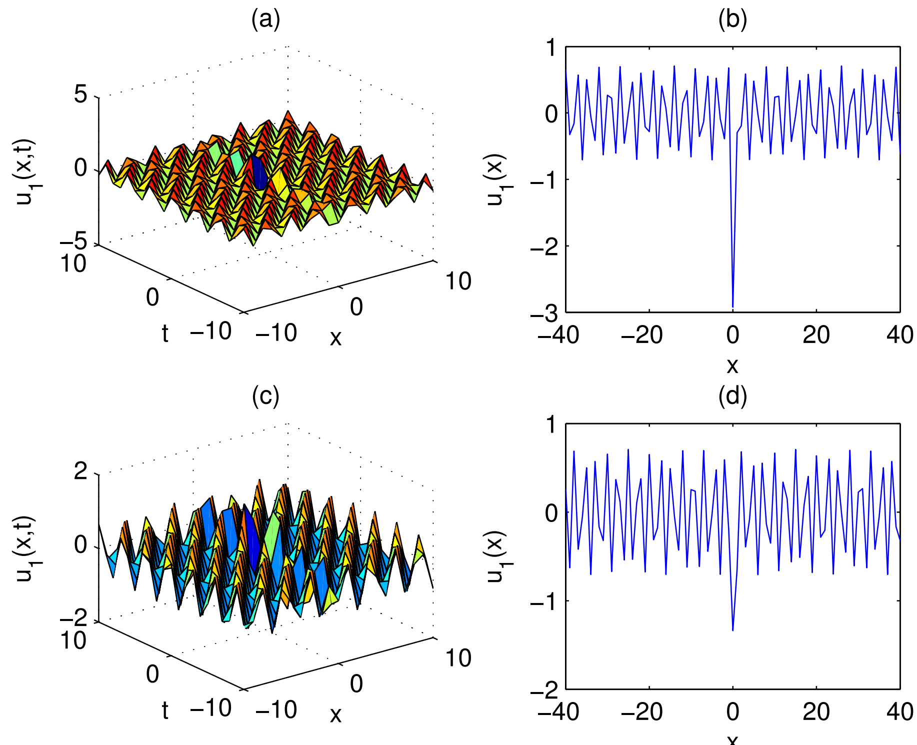

FIGURE 1. Graphs of (a) and (b) real values and (c) and (d) imaginary values of u_1 (3.15) are demonstrated at $a = 1, b = 1, c = 2, B_1 = 1, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (a) and (c) $-10 < x < 10, -10 < t < 10$ and (b) and (d) $-40 < x < 40, t = 1$.

be given. We depict the graph and explain the obtained solutions to the Hamiltonian amplitude equation. In Figures 1-5, we plot three-dimensional graphics of real and imaginary values of (3.15), (3.33), (3.34), and (3.35) respectively, which denote the dynamics of solutions with appropriate parametric selections. We plot three dimensional graphics of Figures 1-5, when $-10 < x < 10, -10 < t < 10$. Solutions u_1, u_{34} of the Hamiltonian amplitude equation represent the exact periodic traveling wave solutions. Also, solutions u_{36}, u_{37}, u_{43} of the Hamiltonian amplitude equation are presented cuspon.

4. Formulation of the variational principle

In this section, we consider the Hamiltonian amplitude equation as the general forth order nonlinear fractional partial differential equation of the form:

$$iu_x + u_{tt} + 2\sigma|u|^2u - \varepsilon u_{xt} = 0, \tag{4.1}$$

where $u(x,t)$ is a complex function, $|u|^2 = u.u^*$ and $*$ denotes a complex conjugate. Substituting $u(x,t) = U(x,t) + iV(x,t)$, where $U(x,t)$ and $V(x,t)$ are real functions of x and t in Equation (4.1), leads to the following coupled nonlinear partial differential equations:

$$U_x + V_{tt} + 2\sigma V(U^2 + V^2) - \varepsilon V_{xt} = 0, \tag{4.2}$$

$$-V_x + U_{tt} + 2\sigma U(U^2 + V^2) - \varepsilon U_{xt} = 0. \tag{4.3}$$

By discussing the existence of a Lagrangian and the invariant variational principle for Equation (4.1) in order to reduce it to a system of two second-order equations, we can express it in the following forms:

$$M(U,V) = U_x + V_{tt} + 2\sigma V(U^2 + V^2) - \varepsilon V_{xt}, \tag{4.4}$$

$$N(U,V) = -V_x + U_{tt} + 2\sigma U(U^2 + V^2) - \varepsilon U_{xt}. \tag{4.5}$$

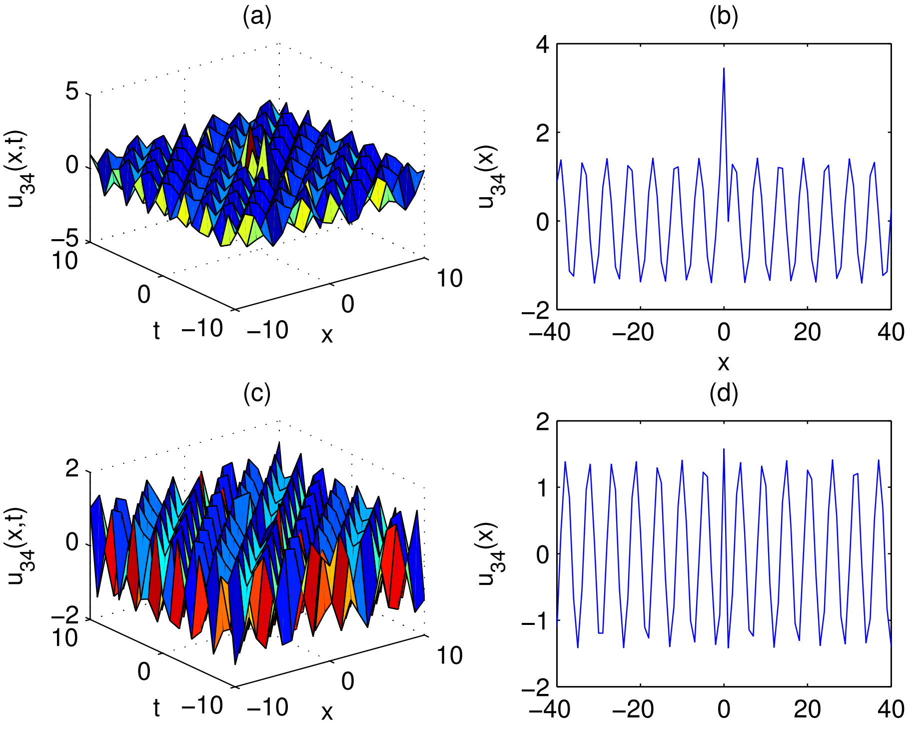

FIGURE 2. Graphs of (a) and (b) real values and (c) and (d) imaginary values of u_{34} (3.33) are demonstrated at $a = 1, b = 1, c = 2, A_1 = 1, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (a) and (c) $-10 < x < 10, -10 < t < 10$ and (b) and (d) $-40 < x < 40, t = 1$.

The coupled nonlinear partial differential Equations (4.4) and (4.5) satisfied the consistency conditions are expressed in [39, 40], then a functional integral $J(U, V)$ can be written down using the formula given by [39, 40], as

$$\begin{aligned} J(U,V) &= \int_{\Upsilon} U \left[\int_0^1 N(\lambda U, \lambda V) d\lambda \right] d\Upsilon + \int_{\Upsilon} V \left[\int_0^1 M(\lambda U, \lambda V) d\lambda \right] d\Upsilon \\ &= \frac{1}{2} \int_{\Upsilon} U \left[-V_x + U_{tt} + 2\sigma U(U^2 + V^2) - \varepsilon U_{xt} \right] d\Upsilon + \frac{1}{2} \int_{\Upsilon} V \left[U_x + V_{tt} + 2\sigma V(U^2 + V^2) - \varepsilon V_{xt} \right] d\Upsilon, \end{aligned} \tag{4.6}$$

where $d\Upsilon = dxdt$. On choosing the boundary on u_x and v_x to be such that the boundary terms vanish, we get the functional integral in the form

$$J(U,V) = \frac{1}{2} \int_{\Upsilon} \left[-UV_x + U(U_{tt} - \varepsilon U_{xt}) + 2\sigma(U^2 + V^2)^2 + VU_x + V(V_{tt} - \varepsilon V_{xt}) \right] d\Upsilon. \tag{4.7}$$

Therefore, the Lagrangian L is given by

$$L(U,V) = \frac{1}{2} \left[-UV_x + U(U_{tt} - \varepsilon U_{xt}) + 2\sigma(U^2 + V^2)^2 + VU_x + V(V_{tt} - \varepsilon V_{xt}) \right]. \tag{4.8}$$

As a necessary check on our calculations, we use the value of L in the Euler-Lagrange equations,

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) = 0, \qquad \frac{\partial L}{\partial v} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial v_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial v_x}\right) = 0, \tag{4.9}$$

which yields the Hamiltonian amplitude equation.

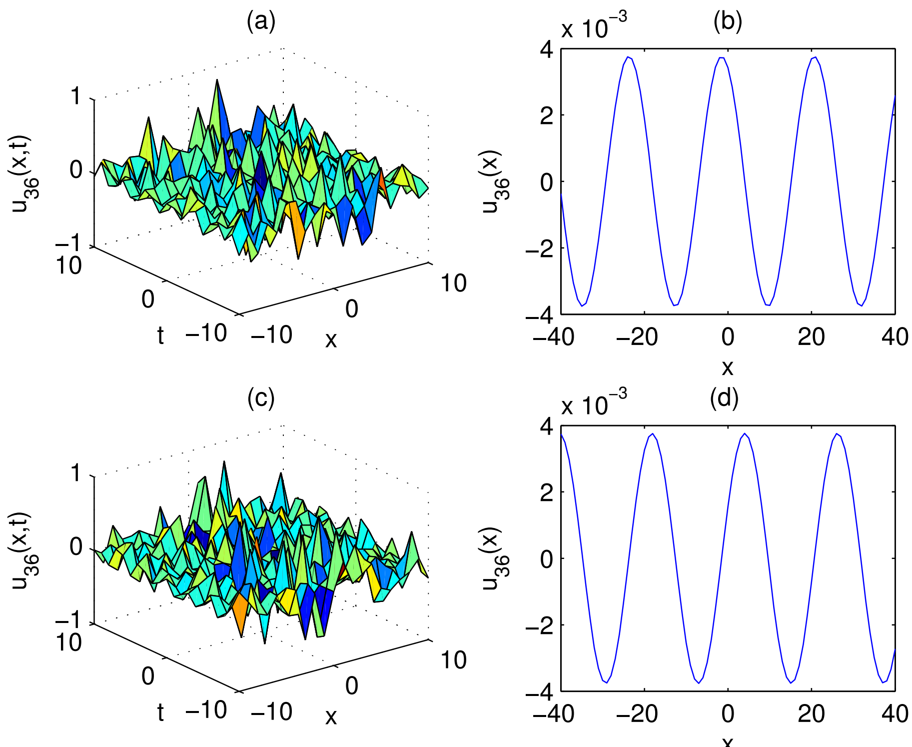

FIGURE 3. Graphs of (a) and (b) real values and (c) and (d) imaginary values of u_{36} (3.34) are demonstrated at $a = 0, b = 2, c = 0, A_1 = 1, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (a) and (c) $-10 < x < 10, -10 < t < 10$ and (b) and (d) $-40 < x < 40, t = 1$.

4.1. **The rectangular box.** Based on linear Jost functions in a single nontrivial variational parameter, we take the box-shaped initial pulse and an ansatz as follows

$$U(x,t) = \begin{cases} \frac{1}{2}\left[\exp(6\pi - x - t) - \exp(-2\pi - x - t)\right], & if\ t > \pi,\ x > \pi, \\ \sinh(2\pi + x + t), & if\ |x| < \pi,\ |t| < \pi, \\ 0, & if\ x < -\pi,\ t < -\pi, \end{cases} \tag{4.10}$$

$$V(x,t) = \begin{cases} 0, & if\ \ t > \pi,\ x > \pi, \\ \sinh(2\pi - x - t), & if\ |x| < \pi,\ |t| < \pi, \\ \frac{1}{2}\left[\exp(6\pi + x + t) - \exp(-2\pi + x + t)\right], & if\ x < -\pi,\ t < -\pi. \end{cases} \tag{4.11}$$

Substituting Eqs. (4.10) and (4.11) into (4.7), one can find the values of the integral L, which determine the Lagrangian according to (4.8),

$$J(U,V) = \int_{-\infty}^{-\pi}\int_{-\infty}^{-\pi} L(U,V)dxdt + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} L(U,V)dxdt + \int_{\pi}^{\infty}\int_{\pi}^{\infty} L(U,V)dxdt, \tag{4.12}$$

$$J(U,V) = 1.056432344 \times 10^{20},$$

where has been considered $\epsilon = 0.01$ and $\sigma = 1$ in (4.8).

4.2. **The two-box potential.** We consider the Jost functions being approximated by a piecewise linear ansatz, which has two variational parameters. The following Jost functions are for two cases:
Case I: **First Set**.

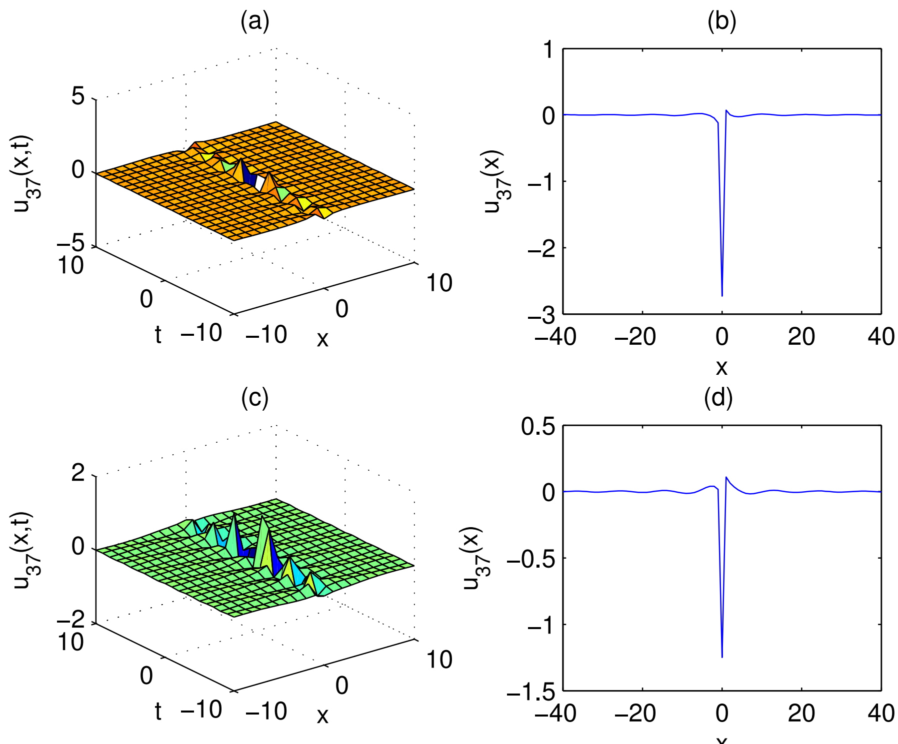

FIGURE 4. Graphs of (a) and (b) real values and (c) and (d) imaginary values of u_{37} (3.34) are demonstrated at $a = 3, b = 4, c = 5, A_1 = 1, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (a) and (c) $-10 < x < 10, -10 < t < 10$ and (b) and (d) $-40 < x < 40, t = 1$.

$$U(x,t) = \begin{cases} \frac{1}{2}\left[\exp(\pi^2(3+2\alpha+\alpha^2) - \pi x - \pi t) - \exp(\pi^2(1-2\alpha-\alpha^2) - \pi x - \pi t)\right], & if\ \ t > \pi,\ x > \pi, \\ \sinh(\pi+\alpha t)(\pi+\alpha x), & if\ 0 < x < \pi,\ 0 < t < \pi, \\ \sinh(\pi+t)(\pi+x), & if\ \ -\pi < x < 0,\ -\pi < t < 0, \\ 0, & if\ x < -\pi,\ t < -\pi, \end{cases} \tag{4.13}$$

$$V(x,t) = \begin{cases} 0, & if\ \ t > \pi,\ x > \pi, \\ \sinh(\pi-t)(\pi-x), & if\ \ -\pi < x < 0,\ -\pi < t < 0, \\ \sinh(\pi-\alpha t)(\pi-\alpha x), & if\ 0 < x < \pi,\ 0 < t < \pi, \\ \frac{1}{2}\left[\exp(\pi^2(1-2\alpha-\alpha^2) + \pi x + \pi t) - \exp(\pi^2(3+2\alpha+\alpha^2) + \pi x + \pi t)\right], & if\ x < -\pi,\ t < -\pi. \end{cases} \tag{4.14}$$

Substituting Eqs. (4.13) and (4.14) into (4.7), one can find the values of the integral L, which determine the Lagrangian according to (4.8),

$$J(U,V) = \int_{-\infty}^{-\pi}\int_{-\infty}^{-\pi} L(U,V)dxdt + \int_{-\pi}^{0}\int_{-\pi}^{0} L(U,V)dxdt + \int_{0}^{\pi}\int_{0}^{\pi} L(U,V)dxdt + \int_{\pi}^{\infty}\int_{\pi}^{\infty} L(U,V)dxdt, \tag{4.15}$$

where

$$\int_{-\infty}^{-\pi}\int_{-\infty}^{-\pi} L(U,V)dxdt = 8.031387692 \times 10^{150}, \qquad \int_{\pi}^{\infty}\int_{\pi}^{\infty} L(U,V)dxdt = 8.031387623 \times 10^{150}, \tag{4.16}$$
$$\int_{-\pi}^{0}\int_{-\pi}^{0} L(U,V)dxdt = 2.121070169 \times 10^{149}, \qquad \int_{0}^{\pi}\int_{0}^{\pi} L(U,V)dxdt = -1.597675814 \times 10^{151}.$$

Thus, one can found $J(U,V)$ as

$$J(U,V) = 2.981241919 \times 10^{149},$$

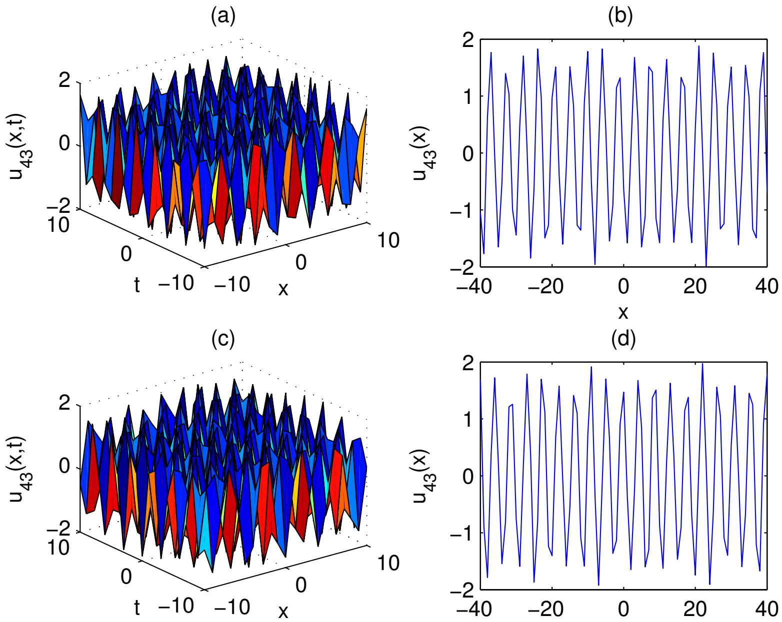

FIGURE 5. Graphs of (a) and (b) real values and (c) and (d) imaginary values of u_{43} (3.35) are demonstrated at $a = 3, b = 4, c = -4, A_1 = 1, \varepsilon = \frac{1}{3}, \beta = 2, \sigma = 1$ and by considering the values (a) and (c) $-10 < x < 10, -10 < t < 10$ and (b) and (d) $-40 < x < 40, t = 1$.

where has been considered $\epsilon = 0.01$, $\sigma = 1$, and $\alpha = 2$ in (4.8).

Case II: **Second Set**.

$$U(x,t) = \begin{cases} \frac{1}{2}\left[\exp(4\pi - x - t + 2\pi\alpha) - \exp(-x - t - 2\pi\alpha)\right], & if \ \ t > \pi, \ x > \pi, \\ \sinh(2\pi + \alpha t + \alpha x), & if \ 0 < x < \pi, \ 0 < t < \pi, \\ \sinh(2\pi + t + x), & if \ \ -\pi < x < 0, \ -\pi < t < 0, \\ 0, & if \ x < -\pi, \ t < -\pi, \end{cases} \tag{4.17}$$

$$V(x,t) = \begin{cases} 0, & if \ \ t > \pi, \ x > \pi, \\ \sinh(2\pi - t - x), & if \ \ -\pi < x < 0, \ -\pi < t < 0, \\ \sinh(2\pi - \alpha t - \alpha x), & if \ 0 < x < \pi, \ 0 < t < \pi, \\ \frac{1}{2}\left[\exp(4\pi + x + t + 2\pi\alpha) - \exp(x + t - 2\pi\alpha)\right], & if \ x < -\pi, \ t < -\pi. \end{cases} \tag{4.18}$$

Substituting Eqs. (4.17) and (4.18) into (4.7), one can find the values of the integral L, which determine the Lagrangian according to (4.8),

$$J(U,V) = \int_{-\infty}^{-\pi}\int_{-\infty}^{-\pi} L(U,V)dxdt + \int_{-\pi}^{0}\int_{-\pi}^{0} L(U,V)dxdt + \int_{0}^{\pi}\int_{0}^{\pi} L(U,V)dxdt + \int_{\pi}^{\infty}\int_{\pi}^{\infty} L(U,V)dxdt, \tag{4.19}$$

$$J(U,V) = 5.429158739 \times 10^{30}, \qquad \epsilon = 0.01, \ \sigma = 1, \ \alpha = 2, \tag{4.20}$$

$$J(U,V) = 2.176960534 \times 10^{9}, \qquad \epsilon = 0.01, \ \sigma = 1, \ \alpha = -2, \tag{4.21}$$

$$J(U,V) = 1.750089295 \times 10^{9}, \qquad \epsilon = 0.01, \ \sigma = 1, \ \alpha = -1.5, \tag{4.22}$$

$$J(U,V) = 2.187852023 \times 10^{25}, \qquad \epsilon = 0.01,\ \sigma = 1,\ \alpha = 1.5, \tag{4.23}$$

where has been considered in (4.8).

4.3. **Tanh functions series.** On the basis of a different ansatz, where we approximate the Jost functions by quadratic polynomials instead of the tanh series of piecewise linear functions. The following Jost functions are for three cases:
Case I: **First Set**.

$$U(x,t) = \tanh(2\pi - x - t), \qquad |x| < 10, \qquad |t| < 10, \tag{4.24}$$

$$V(x,t) = \tanh(2\pi + x + t), \qquad |x| < 10, \qquad |t| < 10. \tag{4.25}$$

Substituting Eqs. (4.24) and (4.25) into (4.7), one can find the values of the integral L, which determine the Lagrangian according to (4.8),

$$J(U,V) = \int_{-10}^{10}\int_{-10}^{10} L(U,V)dxdt = 1371.569402, \tag{4.26}$$

where has been considered $\epsilon = 0.01$ and $\sigma = 1$ in (4.8).
Case II: **Second Set**.

$$U(x,t) = sech^2(2\pi - x - t), \qquad |x| < 1, \qquad |t| < 1, \tag{4.27}$$

$$V(x,t) = sech^2(2\pi + x + t), \qquad |x| < 1, \qquad |t| < 1. \tag{4.28}$$

Substituting Eqs. (4.27) and (4.28) into (4.7), one can find the values of the integral L, which determine the Lagrangian according to (4.8),

$$J(U,V) = \int_{-1}^{1}\int_{-1}^{1} L(U,V)dxdt = 1.318697734 \times 10^{-7}, \tag{4.29}$$

where has been considered $\epsilon = 0.1$ and $\sigma = 1$ in (4.8).
Case III: **Third Set**.

$$U(x,t) = sech(2\pi - x - t)\tanh(2\pi - x - t), \qquad |x| < 1, \qquad |t| < 1, \tag{4.30}$$

$$V(x,t) = sech(2\pi + x + t)\tanh(2\pi + x + t), \qquad |x| < 1, \qquad |t| < 1. \tag{4.31}$$

Substituting Eqs. (4.27) and (4.28) into (4.7), one can find the values of the integral L, which determine the Lagrangian according to (4.8),

$$J(U,V) = \int_{-1}^{1}\int_{-1}^{1} L(U,V)dxdt = 0.0002207618081, \tag{4.32}$$

where has been considered $\epsilon = 0.1$ and $\sigma = 1$ in (4.8).

Remark 4.1. Figures 1-3 show the examples of the Lagrangian $L(x;t)$ with Eq. (4.1). In Figure 1 case (a), by choosing the trial functions (4.10) and (4.11) in the interval $-\pi < x < \pi, -\pi < t < \pi$, in Figure 1 case (b), by choosing the trial functions (4.13) and (4.14) in the interval $0 < x < \pi, 0 < t < \pi$, in Figure 2 case(a), by choosing the trial functions (4.17) and (4.18) in the interval $0 < x < \pi, 0 < t < \pi$, in Figure 2 case (b), by choosing the trial functions (4.24) and (4.25) in the interval $-10 < x < 10, -10 < t < 10$, in Figure 3, by choosing the trial functions (4.27), (4.28), (4.30), and (4.31) in the interval $-1 < x < 1, -1 < t < 1$.

• **Note that**: All the obtained results have been checked with Maple 13 by putting them back into the original equation and found it correct.

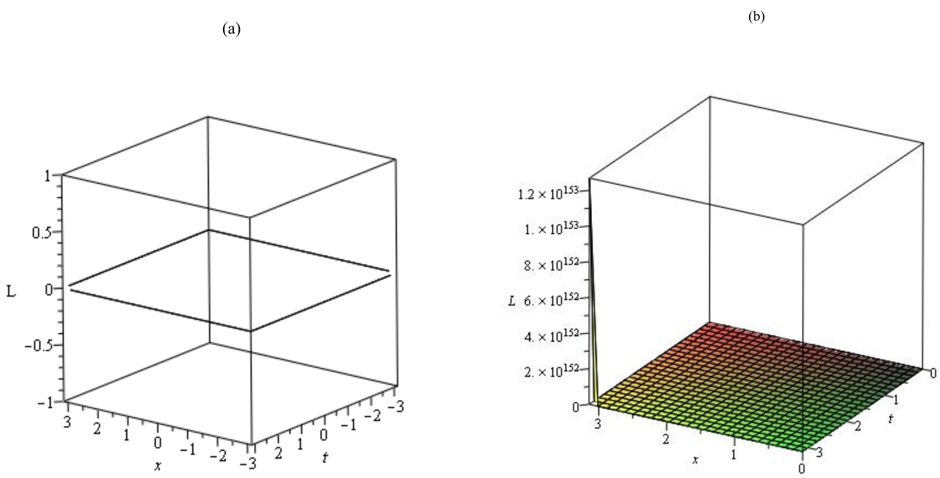

FIGURE 6. (a) Graphs of Lagrangian L by (4.10) and (4.11) for the HAE when $-\pi < x < \pi, -\pi < t < \pi$, (b) Graphs of Lagrangian L by (4.13) and (4.14) for the HAE when $0 < x < \pi, 0 < t < \pi$.

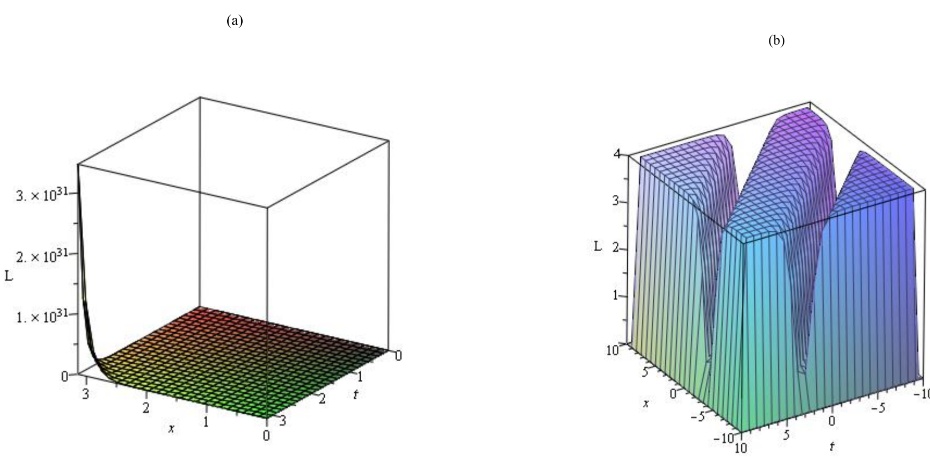

FIGURE 7. (a) Graphs of Lagrangian L by (4.17) and (4.18) for the HAE when $0 < x < \pi, 0 < t < \pi$, (b) Graphs of Lagrangian L by (4.24) and (4.25) for the HAE when $-10 < x < 10, -10 < t < 10$.

5. Conclusions

In this paper, we presented the improved $\tan(\phi(\xi)/2)$-expansion method for solving the Hamiltonian amplitude equation. We extended the ITEM proposed by Manafian et al. [26] to construct new types of soliton wave solutions of nonlinear partial differential equations. The merit of the presented method is finding further solutions to the considered problems, including soliton, periodic, kink, and kink-singular wave solutions. Comparing our new results with other results shows that our results provide further solutions. To the best of our knowledge, the application of the ITEM to the HSE has not been previously submitted to the literature. By using the invariant variational principle, the HSE was transformed into two coupled equations. The approximation solutions of HSE are obtained. By using trial functions, the functional integral and the Lagrangian of the

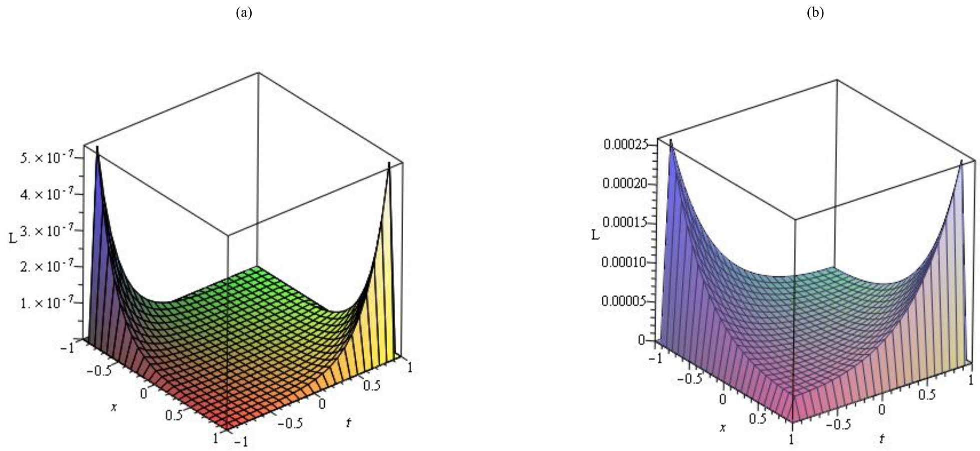

Figure 8. (a) Graphs of Lagrangian L by (4.27) and (4.28) for the HAE when $-1 < x < 1, -1 < t < 1$, (b) Graphs of Lagrangian L by (4.30) and (4.31) for the HAE when $-1 < x < 1, -1 < t < 1$.

system without loss are found. Moreover, the general case for the two-box potential can be obtained in the basis of a different ansatz, where we approximated the Jost function by a series in the tanh function method instead of the piece-wise linear function one. It can be concluded that these methods are very powerful and efficient techniques in finding exact solutions for wide classes of problems, particularly in mechanical engineering.

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