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Application of $tan(\phi/2)$-expansion method for solving the fractional Biswas-Milovic equation for Kerr law nonlinearity

Dmitry Fugarov[1,∗], Alexey Dengaev[2], Ilya Drozdov[2], Vladimir Shishulin[2], and Anastasiya Ostrovskaya[3]
[1]Don State Technical University, Rostov-on-Don, Russia.
[2]Gubkin Russian State University of Oil and Gas, Moscow, Russia.
[3]Kuban State University, Krasnodar, Russia.

Abstract

In this paper, the improved $\tan(\Phi(\xi)/2)$-expansion method (ITEM) is proposed to obtain the fractional Biswas-Milovic equation. The exact particular solutions contain four types: hyperbolic function solution, trigonometric function solution, exponential solution, and rational solution. We obtained further solutions compared with other methods, such as [2]. Recently, this method has been developed for searching exact travelling wave solutions of nonlinear partial differential equations. These solutions might play an important role in nonlinear optics and physics. It is shown that this method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving problems in nonlinear optics.

Keywords. Improved $\tan(\Phi(\xi)/2)$-expansion method, Fractional Biswas-Milovic equation, Exact soliton solution.
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1. Introduction

In the recent decade, fractional nonlinear differential equations have been demonstrated in numerous applications seemingly different fields of engineering sciences, physics, finance, applied mathematics, and others [9, 11, 19, 42, 44]. Different researchers worked on nonlinear fractional equations. In this paper, we consider the fractional Biswas-Milovic equation [2, 7] as follows

$$iD_t^{\alpha}q^n + \lambda D_x^{2\beta}q^n + \mu F(|q|^2)q^n = 0, \tag{1.1}$$

where $\lambda\mu > 0$, $0 < \alpha \le 1$, $0 < \beta \le 1$, $n \ge 1$ and $q = q(x,t)$ is a complex valued function. The coefficients λ and μ represent the coefficients of group velocity dispersion and nonlinearity, respectively. In Eq. (1.1), F is a real-valued algebraic function and to satisfy the necessary condition of having smoothness of the complex function $F(|q|^2)q$, the function $F(|q|^2)q$ is considered to be k times continuously differentiable [6, 27]. A real understanding of the dynamics of optical solitons with a generalized flavor is considered by using the BM equation. Also, the governing equation is of special interest in the nonlinear fiber optics community [27]. For further information on the dynamics of solitons in optical fibers, please refer to ([5]–[45]). To solve the BM equation with variable physical properties, different methods have been proposed by authors ([3, 6, 17, 18, 21, 43]). The nonlinear partial differential equations play a key role in describing key scientific phenomena. In fact, it has been discovered that many models in mathematics and physics are described by nonlinear partial differential equations. With the rapid development of nonlinear sciences based on computer algebraic systems, many effective methods have been presented, such as the homotopy analysis method [13], the variational iteration method [15], the Adomian decomposition method [20], the homotopy perturbation method [12], the tanh-coth method [28], the Exp-function method [14, 23, 29], the G'/G-expansion method [32, 33], the homogeneous balance method [49], the formal linearization method [40], and so on.

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∗ Corresponding author. Email: dmitryfugarov@yandex.ru.

In this paper, we have two goals. First, we introduce a general form of the ITEM [1, 24, 25, 30, 31, 34, 35, 46, 48], which is a new method. Next, we obtain the exact solutions of the Biswas-Milovic equation for one type of nonlinearity by the aforementioned method.

Authors of [10] explained the generalized fifth-order KdV-like equation with prime number $p = 3$ via a generalized bilinear differential operator. N-lump was investigated to the variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [36]. Applications of $tan(\phi/2)$-expansion method for the Biswas-Milovic equation [37], the Gerdjikov-Ivanov model [38], the Kundu-Eckhaus equation [39], and the fifth-order integrable equations [22] were studied. Lump solutions were analyzed to the fractional generalized CBS-BK equation [47] and the (3+1)-D Burger system [16]. The approximations of a one-dimensional hyperbolic equation with non-local integral conditions were constructed by the reduced differential transform method [41]. The generalized Hirota bilinear strategy by the prime number was used for the (2+1)-dimensional generalized fifth-order KdV-like equation [26]. The traveling wave solutions and analytical treatment of the simplified MCH equation and the combined KdV-mKdV equations were studied [4].
The outline of this paper is organized as follows:
In section 2, we describe the ITEM. In section 3, we apply the mathematical analysis of the Biswas-Milovic equation. Section 4 will be further analyzed the Kerr law nonlinearity. Also, the conclusion is given in section 5.

2. Description of the ITEM

The ITEM is well-known analytical method which was improved and developed by Manafian.
Step 1. We suppose that the given nonlinear partial differential equation for u(x, t) is in the form

$$\mathcal{N}(u, u_x, u_t, u_{xx}, u_{tt}, ...) = 0, \tag{2.1}$$

which can be converted to an ODE

$$\mathcal{Q}(u, u', -\mu u', u'', \mu^2 u'', ...) = 0, \tag{2.2}$$

by the transformation $\xi = x - \mu t$ is the wave variable. Also, μ is constant to be determined later.
Step 2. Suppose the traveling wave solution of Eq. (2.2) can be expressed as follows:

$$u(\xi) = S(\phi) = \sum_{k=-m}^{m} A_k \left[p + \tan(\phi/2)\right]^k, \tag{2.3}$$

where $A_k (0 \le k \le m)$ and $A_{-k} = B_k (1 \le k \le m)$ are constants to be determined, such that $A_m \neq 0, B_m \neq 0$, and $\phi = \phi(\xi)$ satisfy the following ordinary differential equation:

$$\phi'(\xi) = a \sin(\phi(\xi)) + b \cos(\phi(\xi)) + c. \tag{2.4}$$

We will consider the following special solutions of Eq. (2.4):
Family 1: **When** $\Delta = a^2 + b^2 - c^2 < 0$ **and** $b - c \neq 0$**,** then
$\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\bar{\xi}\right)\right]$.
Family 2: **When** $\Delta = a^2 + b^2 - c^2 > 0$ **and** $b - c \neq 0$**,** then
$\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\bar{\xi}\right)\right]$.
Family 3: **When** $\Delta = a^2 + b^2 - c^2 > 0$**,** $b \neq 0$ **and** $c = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\frac{a}{b} + \frac{\sqrt{b^2+a^2}}{b}\tanh\left(\frac{\sqrt{b^2+a^2}}{2}\bar{\xi}\right)\right]$.
Family 4: **When** $\Delta = a^2 + b^2 - c^2 < 0$**,** $c \neq 0$ **and** $b = 0$, then $\phi(\xi) = 2\tan^{-1}\left[-\frac{a}{c} + \frac{\sqrt{c^2-a^2}}{c}\tan\left(\frac{\sqrt{c^2-a^2}}{2}\bar{\xi}\right)\right]$.
Family 5: **When** $\Delta = a^2 + b^2 - c^2 > 0$**,** $b - c \neq 0$ **and** $a = 0$, then $\phi(\xi) = 2\tan^{-1}\left[\sqrt{\frac{b+c}{b-c}}\tanh\left(\frac{\sqrt{b^2-c^2}}{2}\bar{\xi}\right)\right]$.
Family 6: **When** $a = 0$ **and** $c = 0$, then $\phi(\xi) = \tan^{-1}\left[\frac{e^{2b\bar{\xi}}-1}{e^{2b\bar{\xi}}+1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}}+1}\right]$.
Family 7: **When** $b = 0$ **and** $c = 0$, then $\phi(\xi) = \tan^{-1}\left[\frac{2e^{a\bar{\xi}}}{e^{2a\bar{\xi}}+1}, \frac{e^{2a\bar{\xi}}-1}{e^{2a\bar{\xi}}+1}\right]$.
Family 8: **When** $a^2 + b^2 = c^2$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{a\bar{\xi}+2}{(b-c)\bar{\xi}}\right]$.
Family 9: **When** $a = b = c = ka$, then $\phi(\xi) = 2\tan^{-1}\left[e^{ka\bar{\xi}} - 1\right]$.

Family 10: **When** $a = c = ka$ **and** b = −ka**,** then $\phi(\xi) = -2\tan^{-1}\left[\frac{e^{ka\bar{\xi}}}{-1+e^{ka\bar{\xi}}}\right]$.
Family 11: **When** $c = a$**,** then $\phi(\xi) = -2\tan^{-1}\left[\frac{(a+b)e^{b\bar{\xi}}-1}{(a-b)e^{b\bar{\xi}}-1}\right]$.
Family 12: **When** $a = c$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{(b+c)e^{b\bar{\xi}}+1}{(b-c)e^{b\bar{\xi}}-1}\right]$.
Family 13: **When** $c = -a$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{e^{b\bar{\xi}}+b-a}{e^{b\bar{\xi}}-b-a}\right]$.
Family 14: **When** $b = -c$**,** then $\phi(\xi) = 2\tan^{-1}\left[\frac{ae^{a\bar{\xi}}}{1-ce^{a\bar{\xi}}}\right]$.
Family 15: **When** $b = 0$ **and** a = c**,** then $\phi(\xi) = -2\tan^{-1}\left[\frac{c\bar{\xi}+2}{c\bar{\xi}}\right]$.
Family 16: **When** $a = 0$ **and** $b = c$**,** then $\phi(\xi) = 2\tan^{-1}\left[c\bar{\xi}\right]$.
Family 17: **When** $a = 0$ **and** $b = -c$**,** then $\phi(\xi) = -2\tan^{-1}\left[\frac{1}{c\bar{\xi}}\right]$.
Family 18: **When** $a = 0$ **and** $b = 0$**,** then $\phi(\xi) = c\xi + C$.
Family 19: **When** $b = c$ then $\phi(\xi) = 2\tan^{-1}\left[\frac{e^{a\bar{\xi}}-c}{a}\right]$, where $\bar{\xi} = \xi + C, p, A_0, A_k, B_k (k = 1, 2, ..., m), a, b$ and c are constants to be determined later.
Step 3. Determine m. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest-order nonlinear term(s) in Eq. (2.2). However, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in Eq. (2.2).
Step 4. Substituting (2.3) into Eq. (2.2) with the value of m obtained in Step 2. Collecting the coefficients of $\tan(\phi/2)^k$, $\cot(\phi/2)^k (k = 0, 1, 2, ...)$, then setting each coefficient to zero, we can get a set of over-determined equations for $A_0, A_k, B_k (k = 1, 2, ..., m)$, a, b, c, and p with the aid of symbolic computation Maple.
Step 5. Solving the algebraic equations in Step 3, then substituting $A_0, A_1, B_1, ..., A_m, B_m, \mu$, and p in (2.3).

3. Mathematical analysis of the fractional BM equation

In this section, we consider the dimensionless form of the fractional BM equation to be studied in this paper which is given in

$$iD_t^{\alpha} q^n + \lambda D_x^{2\beta} q^n + \mu F(|q|^2) q^n = 0, \tag{3.1}$$

where $\lambda\mu > 0$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $n \geq 1$ and $q = q(x, t)$ is a complex valued function. The coefficients λ and μ represent the coefficients of group velocity dispersion and nonlinearity, respectively. Thus, if $n = 1$, Eq. (3.1) collapses to NLSE that arises in nonlinear optics, fluid dynamics, plasma physics, mathematical biology, and several other areas. In this paper, we search for the stationary solution to (3.1). The starting hypothesis is taken to be

$$q(x, t) = u(\xi)\exp(i\theta), \qquad \xi = x - \frac{\eta t^{\alpha}}{\Gamma(1+\alpha)}, \qquad \theta = \frac{st^{\beta}}{\Gamma(1+\beta)} + \frac{rt^{\alpha}}{\Gamma(1+\alpha)}, \tag{3.2}$$

where α represents the soliton wave number, β is the soliton frequency, and γ is the phase constant. Thus, from (3.1)

$$D_t^{\alpha} q^n = nu^{n-1}(-\eta u' + iru)e^{in\theta}, \tag{3.3}$$

$$D_x^{\beta} q^n = nu^{n-1}(u' + isu)e^{in\theta}, \tag{3.4}$$

$$D_x^{2\beta} q^n = nu^{n-2}[(n-1)(u' + isu)^2 + u(u'' + 2isu' - s^2 u)]e^{in\theta}. \tag{3.5}$$

Inserting (3.3) into (3.5) separating to real and imaginary parts, the results are
Real part:

$$-n(r + \lambda ns^2)u^n + \lambda n(n-1)u^{n-2}(u')^2 + \lambda nu^{n-1}u'' + \mu F(u^2)u^n = 0. \tag{3.6}$$

Imaginary part:

$$iu'u^{n-1}[-\eta n + 2\lambda n^2 s] = 0, \qquad \Rightarrow \eta = 2\lambda ns. \tag{3.7}$$

4. Kerr law nonlinearity

This section will be further analyzed the Kerr law nonlinearity via $tan(\phi/2)$-expansion method.

4.1. **The Case n=1 case for the fractional BME.** We start our study by assuming $\mathbf{n}=\mathbf{1}$ in (3.6), therefore we have

$$-(\mathrm{r}+\lambda \mathrm{s}^2)\mathrm{u}(\xi)+\lambda \mathrm{u}''(\xi)+\mu \mathrm{F}(\mathrm{u}^2(\xi))\mathrm{u}(\xi)=0. \tag{4.1}$$

In the presence of perturbation terms, the fractional BME with Kerr law nonlinearity ($\mathrm{F(w)=w}$) is given by

$$-(\mathrm{r}+\lambda \mathrm{s}^2)\mathrm{u}(\xi)+\lambda \mathrm{u}''(\xi)+\mu \mathrm{u}^3(\xi)=0. \tag{4.2}$$

The next step is to expand the unknowns $u(\xi)$ in power series in terms of $p+\tan(\phi/2)$,

$$u(\xi)=\sum_{k=-m}^{m} A_k\left[p+\tan(\phi(\xi)/2)\right]^k, \tag{4.3}$$

which $A_{-k}=B_k$. In order to determine value of m, we balance the linear term of the highest order u'' with the highest order nonlinear term u^3 in Eq. (4.2) we get

$$u(\xi)=A_m\left(\tan(\phi(\xi)/2)\right)^m+\ldots, \tag{4.4}$$

$$u^3(\xi)=A_m^3\left(\tan(\phi(\xi)/2)\right)^{3m}+\ldots, \tag{4.5}$$

$$\frac{du(\xi)}{d\xi}=\frac{m(c-b)}{2}A_m\left(\tan(\phi(\xi)/2)\right)^{m+1}+\ldots, \tag{4.6}$$

$$\frac{d^2u(\xi)}{d\xi^2}=\frac{m(m+1)(c-b)^2}{2}A_m\left(\tan(\phi(\xi)/2)\right)^{m+2}+\ldots. \tag{4.7}$$

By considering the homogeneous balance principle between the highest order derivatives u'' and nonlinear terms u^3, we obtain $m+2=3m$, then $m=1$. Suppose that the solutions for Eq. (4.2) can be expressed in the following form

$$u(\xi)=\sum_{k=-1}^{1} A_k(p+\tan(\phi/2))^k, \tag{4.8}$$

Substituting (4.8) and (2.4) into Eq. (4.2) and collecting all terms with the same order of $tan\left(\Phi(\xi)/2\right)$ together, and setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for a,b,c,k,w,A_0,A_1, and B_1 as follows:
Coefficients of $Y=\tan(\phi(\xi)/2)$

$$\begin{aligned}
&Y^0:\lambda(b+c)(B_1b-B_1pa+A_1p^3a+B_1c)+2(B_1+A_1p^2+pA_0)\\
&\quad(\mu A_1^2p^4+2\mu A_0p^3A_1+2p^2\mu B_1A_1-p^2\lambda s^2-p^2r+\mu A_0^2p^2+2\mu A_0B_1p+\mu B_1^2)=0,\\
&Y^1:\lambda[p(2a^2+c^2-b^2)(A_1p^2-B_1)+3a(A_1p^2+B_1)(b+c)]+4p(B_1+2A_1p^2)(3\mu B_1A_1-\lambda s^2+3\mu A_0^2-r)\\
&\quad+12\mu A_1^3p^5+30\mu A_0A_1^2p^4+6A_0p^2(\mu A_0^2-r+6\mu B_1A_1-\lambda s^2)+6\mu A_0B_1^2=0,\\
&Y^2:-3ap\lambda(p^2A_1b-p^2A_1c-A_1b+B_1c-B_1b-A_1c)+\lambda(-b^2+c^2+2a^2)(B_1+3A_1p^2)+30\mu A_1^3p^4\\
&\quad+6A_0p(10p^2\mu A_1^2+\mu A_0^2-r+6\mu B_1A_1-\lambda s^2)+2(6A_1p^2+B_1)(3\mu B_1A_1+3\mu A_0^2-\lambda s^2-r)=0,\\
&Y^3:p\lambda[A_1c^2(p^2+3)+A_1b^2(p^2-3)-2p^2A_1bc-B_1b^2+2B_1bc-B_1c^2+6A_1a^2]\\
&\quad-a\lambda(9A_1bp^2-9A_1cp^2-A_1b-A_1c-B_1c+B_1b)\\
&\quad+8A_1p(5\mu A_1^2p^2+3\mu B_1A_1+3\mu A_0^2+\lambda s^2-r)+2A_0(\mu A_0^2+30\mu A_1^2p^2+6\mu B_1A_1-r-\lambda s^2)=0,\\
&Y^4:3p^2\lambda A_1(b-c)^2+\lambda A_1(-9abp+9acp+2a^2-b^2+c^2)+30\mu A_1^3p^2\\
&\quad+2A_1(3\mu B_1A_1+15A_1\mu A_0p+3\mu A_0^2-r-\lambda s^2)=0,\\
&Y^5:-3\lambda A_1(b-c)(-pb+pc+a)+6\mu A_1^2(2pA_1+A_0)=0,
\end{aligned}$$

$$Y^6 : A_1(2\mu A_1^2 + \lambda c^2 + \lambda b^2 - 2\lambda cb) = 0. \tag{4.9}$$

Solving the set of algebraic equations using Maple, we get the following results:

Case I

$$a = a, \quad b = b, \quad c = c, \quad p = p, \quad \Delta = a^2 + b^2 - c^2, \quad A_0 = (a + p(b - c))\sqrt{\frac{-\lambda}{2\mu}},$$

$$A_1 = (b - c)\sqrt{\frac{-\lambda}{2\mu}}, \quad B_1 = 0, \quad s = s, \quad r = -\frac{\lambda}{2}(2s^2 + \Delta), \quad u(\xi) = A_0 + A_1\left[p + \tan(\phi(\xi)/2)\right]. \tag{4.10}$$

By using of (4.10) and Families 1, 2, 6, 8, 12, and 15, respectively, can be written as

$$u_1(\xi) = (a + 2p(b - c))\sqrt{\frac{-\lambda}{2\mu}} + \sqrt{\frac{-\lambda}{2\mu}}\left[a - \sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right], \tag{4.11}$$

$$u_2(\xi) = (a + 2p(b - c))\sqrt{\frac{-\lambda}{2\mu}} + \sqrt{\frac{-\lambda}{2\mu}}\left[a + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right],$$

$$u_3(\xi) = 2pb\sqrt{\frac{-\lambda}{2\mu}} + b\sqrt{\frac{-\lambda}{2\mu}}\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}, \frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}\right]\right),$$

$$u_4(\xi) = (a + 2p(b - c))\sqrt{\frac{-\lambda}{2\mu}} + \sqrt{\frac{-\lambda}{2\mu}}\left[\frac{a(\xi + C) + 2}{(\xi + C)}\right],$$

$$u_5(\xi) = (c + 2p(b - c))\sqrt{\frac{-\lambda}{2\mu}} + (b - c)\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{(b + c)e^{b(\xi+C)} + 1}{(b - c)e^{b(\xi+C)} - 1}\right],$$

$$u_6(\xi) = (c - 2pc)\sqrt{\frac{-\lambda}{2\mu}} - \sqrt{\frac{-\lambda}{2\mu}}\left[\frac{c(\xi + C) + 2}{(\xi + C)}\right],$$

where $q(x, t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda st^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+\Delta)t^\alpha}{2\Gamma(1+\alpha)}$.

Case II

$$a = 0, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = 0, \quad A_1 = (b - c)\sqrt{\frac{-\lambda}{2\mu}}, \tag{4.12}$$

$$B_1 = 0, \quad s = s, \quad r = -\frac{\lambda}{2}(2s^2 + b^2 - c^2), \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right].$$

By using of (4.12) and Families 5, 6, 11, and 17, respectively, can be written as

$$u_7(\xi) = \sqrt{\frac{-\lambda(b^2 - c^2)}{2\mu}}\tanh\left(\frac{\sqrt{b^2 - c^2}}{2}(\xi + C)\right), \tag{4.13}$$

$$u_8(\xi) = b\sqrt{\frac{-\lambda}{2\mu}}\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)} + 1}, \frac{e^{2b(\xi+C)} - 1}{e^{2b(\xi+C)} + 1}\right]\right),$$

$$u_9(\xi) = b\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{be^{b(\xi+C)} - 1}{be^{b(\xi+C)} + 1}\right], \quad u_{10}(\xi) = 2\sqrt{\frac{-\lambda}{2\mu}}\frac{1}{\xi + C},$$

where $q(x, t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda st^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+b^2-c^2)t^\alpha}{2\Gamma(1+\alpha)}$.

Case III

$$a = a, \quad b = b, \quad c = c, \quad p = p, \quad \Delta = a^2 + b^2 - c^2, \quad A_0 = (a + p(b - c))\sqrt{\frac{-\lambda}{2\mu}}, \quad A_1 = 0, \tag{4.14}$$

$$B_1 = -(p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}, \quad s=s, \quad r=-\frac{\lambda}{2}(2s^2+\Delta), \quad u(\xi)=A_0+B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.14) and Families 1, 2, 6, 8, 12, and 15, respectively, can be written as

$$u_{11}(\xi) = (a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{a}{b-c}-\frac{\sqrt{-\Delta}}{b-c}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi\right)\right]^{-1}, \quad (4.15)$$

$$u_{12}(\xi) = (a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{a}{b-c}+\frac{\sqrt{\Delta}}{b-c}\tanh\left(\frac{\sqrt{\Delta}}{2}\xi\right)\right]^{-1},$$

$$u_{13}(\xi) = pb\sqrt{\frac{-\lambda}{2\mu}} - b(p^2-1)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right)\right]^{-1},$$

$$u_{14}(\xi) = (a+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2ap-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{a(\xi+C)+2}{(b-c)(\xi+C)}\right]^{-1},$$

$$u_{15}(\xi) = (c+p(b-c))\sqrt{\frac{-\lambda}{2\mu}} - (p^2(b-c)+2cp-b-c)\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{16}(\xi) = c(1-p)\sqrt{\frac{-\lambda}{2\mu}} + c(p-1)^2\sqrt{\frac{-\lambda}{2\mu}}\left[p+\frac{c(\xi+C)+2}{c(\xi+C)}\right]^{-1},$$

where $q(x,t)=u(\xi)e^{i\theta}$, $\xi = x-\frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+\Delta)t^\alpha}{2\Gamma(1+\alpha)}$.

Case IV

$$a=a, \quad b=c, \quad c=c, \quad p=p, \quad A_0 = a\sqrt{\frac{-\lambda}{2\mu}}, \quad A_1=0, \quad (4.16)$$

$$B_1 = (ap-c)\sqrt{\frac{-2\lambda}{\mu}}, \quad s=s, \quad r=-\frac{\lambda}{2}(2s^2+a^2), \quad u(\xi)=A_0+B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.16) and Families 7, 13, and 16, respectively, can be written as

$$u_{17}(\xi) = a\sqrt{\frac{-\lambda}{2\mu}} + (ap-c)\sqrt{\frac{-2\lambda}{\mu}}\left[p+\tan\left(\frac{1}{2}\arctan\left[\frac{2e^{a(\xi+C)}}{e^{2a(\xi+C)}+1}, \frac{e^{2a(\xi+C)}-1}{e^{2a(\xi+C)}+1}\right]\right)\right]^{-1}, \quad (4.17)$$

$$u_{18}(\xi) = -c\sqrt{\frac{-\lambda}{2\mu}} - c(p+1)\sqrt{\frac{-2\lambda}{\mu}}\left[\frac{e^{b(\xi+C)}+2c}{e^{b(\xi+C)}}\right]^{-1}, \qquad u_{19}(\xi) = -c\sqrt{\frac{-2\lambda}{\mu}}\frac{1}{p+c(\xi+C)},$$

where $q(x,t)=u(\xi)e^{i\theta}$, $\xi = x-\frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+a^2)t^\alpha}{2\Gamma(1+\alpha)}$.

Case V

$$a=a, \quad b=b, \quad c=c, \quad p=-\frac{a}{b-c}, \quad A_0=0, \quad A_1=0, \quad B_1=[p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}, \quad (4.18)$$

$$\Delta = a^2+b^2-c^2, \quad s=s, \quad r=-\frac{\lambda}{2}(2s^2+\Delta), \quad u(\xi)=B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.14) and Families 1, 2, 6, 8, 12, and **15**, respectively, can be written as

$$u_{19}(\xi) = -[p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\frac{b-c}{\sqrt{-\Delta}}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi\right), \quad (4.19)$$

$$u_{20}(\xi) = [p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\frac{b-c}{\sqrt{\Delta}}\coth\left(\frac{\sqrt{\Delta}}{2}\xi\right),$$

$$u_{21}(\xi) = b\sqrt{\frac{-\lambda}{2\mu}}\cot\left(\frac{1}{2}\arctan\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right),$$

$$u_{22}(\xi) = [p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\left[-\frac{a}{b-c}+\frac{a(\xi+C)+2}{(b-c)(\xi+C)}\right]^{-1},$$

$$u_{23}(\xi) = [p^2(b-c)+b+c]\sqrt{\frac{-\lambda}{2\mu}}\left[-\frac{c}{b-c}+\frac{(b+c)e^{b(\xi+C)}+1}{(b-c)e^{b(\xi+C)}-1}\right]^{-1},$$

$$u_{24}(\xi) = [-p^2c+c]\sqrt{\frac{-\lambda}{2\mu}}\left[-1+\frac{c(\xi+C)+2}{c(\xi+C)}\right]^{-1},$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+\Delta)t^\alpha}{2\Gamma(1+\alpha)}$.

Case VI

$$a = 0, \quad b = \sqrt{-\frac{\mu}{2\lambda}}B_1, \quad c = \sqrt{-\frac{\mu}{2\lambda}}B_1, \quad p = p, \quad A_0 = 0, \quad A_1 = 0, \tag{4.20}$$

$$B_1 = B_1, \quad s = s, \quad r = -\lambda s^2, \quad u(\xi) = B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.20) and Family 16, we can write

$$u_{25}(\xi) = \frac{B_1}{c(\xi+C)}, \tag{4.21}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} - \frac{\lambda s^2 t^\alpha}{\Gamma(1+\alpha)}$.

Case VII

$$a = 0, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = 0, \quad A_1 = (b-c)\sqrt{\frac{-\lambda}{2\mu}}, \quad B_1 = (b+c)\sqrt{\frac{-\lambda}{2\mu}}, \tag{4.22}$$

$$s = s, \quad r = -\frac{\lambda}{2}(2s^2+4(b^2-c^2)), \quad u(\xi) = A_1\left[p+\tan(\phi(\xi)/2)\right] + B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.22) and Families 5, 6, and **11**, respectively, can be written as

$$u_{26}(\xi) = \sqrt{\frac{-\lambda}{2\mu}}\sqrt{b^2-c^2}\left[\tanh\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)+\coth\left(\frac{\sqrt{b^2-c^2}}{2}(\xi+C)\right)\right], \tag{4.23}$$

$$u_{27}(\xi) = b\sqrt{\frac{-\lambda}{2\mu}}\left[\tan\left(\frac{1}{2}\tan^{-1}\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right)+\cot\left(\frac{1}{2}\tan^{-1}\left[\frac{2e^{b(\xi+C)}}{e^{2b(\xi+C)}+1}, \frac{e^{2b(\xi+C)}-1}{e^{2b(\xi+C)}+1}\right]\right)\right],$$

$$u_{28}(\xi) = 2b\sqrt{\frac{-\lambda}{2\mu}}\left[\frac{b^2e^{2b(\xi+C)}+1}{b^2e^{2b(\xi+C)}-1}\right],$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda s t^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{s t^\beta}{\Gamma(1+\beta)} - \frac{\lambda(2s^2+4(b^2-c^2))t^\alpha}{2\Gamma(1+\alpha)}$.

Case VIII

$$a = pA_1\sqrt{-\frac{2\mu}{\lambda}}, \quad b = -(A_1p^2-A_1+B_1)\sqrt{-\frac{\mu}{2\lambda}}, \quad c = (A_1p^2+A_1+B_1)\sqrt{-\frac{\mu}{2\lambda}}, \tag{4.24}$$

$$p = p, \quad A_0 = 0, \quad A_1 = A_1,$$

$$B_1 = B_1, \quad s = s, \quad r = 2\mu A_1B_1 - \lambda s^2, \quad u(\xi) = A_1\left[p+\tan(\phi(\xi)/2)\right] + B_1\left[p+\tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.24) and Families 1 and **2**, respectively, can be written as

$$u_{29}(\xi) = -\frac{2\lambda}{\mu}\sqrt{A_1B_1}\tan\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right) - \frac{\mu}{2\lambda}\sqrt{A_1B_1}\cot\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right), \tag{4.25}$$

$$u_{30}(\xi) = \frac{2\lambda}{\mu}\sqrt{-A_1B_1}\tanh\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right) + \frac{\mu}{2\lambda}\sqrt{-A_1B_1}\cot\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right),$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda st^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{(2\mu A_1B_1 - \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

Case IX

$$a = pA_1\sqrt{-\frac{2\mu}{\lambda}}, \quad b = -(A_1p^2 - A_1 - B_1)\sqrt{-\frac{\mu}{2\lambda}},$$
$$c = (A_1p^2 + A_1 - B_1)\sqrt{-\frac{\mu}{2\lambda}}, \quad p = p, \quad A_0 = 0, \quad A_1 = A_1, \tag{4.26}$$
$$B_1 = B_1, \quad s = s, \quad r = 4\mu A_1B_1 - \lambda s^2, \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right] + B_1\left[p + \tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.26) and Families 1 and **2**, respectively, can be written as

$$u_{31}(\xi) = -\frac{2\lambda}{\mu}\sqrt{-A_1B_1}\tan\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right) - \frac{\mu}{2\lambda}\sqrt{-A_1B_1}\cot\left(\frac{1}{2}\sqrt{\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right), \tag{4.27}$$

$$u_{32}(\xi) = \frac{2\lambda}{\mu}\sqrt{A_1B_1}\tanh\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right) + \frac{\mu}{2\lambda}\sqrt{A_1B_1}\cot\left(\frac{1}{2}\sqrt{-\frac{2\mu A_1B_1}{\lambda}}(\xi + C)\right),$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda st^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{(4\mu A_1B_1 - \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

Case X

$$a = -iB_1\sqrt{-\frac{\mu}{2\lambda}}, \quad b = 0, \quad c = B_1\sqrt{-\frac{\mu}{2\lambda}}, \quad p = i, \quad A_0 = 0, \quad A_1 = -\frac{1}{2}B_1, \tag{4.28}$$
$$B_1 = B_1, \quad s = s, \quad r = -2\mu B_1^2 - \lambda s^2, \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right] + B_1\left[p + \tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.28) and Family 1, we can write

$$u_{33}(\xi) = -\frac{\sqrt{2}}{2}B_1\tan\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi + C)\right) + \frac{\sqrt{2}}{2}B_1\cot\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi + C)\right), \tag{4.29}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda st^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{(2\mu B_1^2 + \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

Case XI

$$a = -iB_1\sqrt{-\frac{\mu}{2\lambda}}, \quad b = 0, \quad c = B_1\sqrt{-\frac{\mu}{2\lambda}}, \quad p = i, \quad A_0 = 0, \quad A_1 = -\frac{1}{2}B_1, \tag{4.30}$$
$$B_1 = B_1, \quad s = s, \quad r = -2\mu B_1^2 - \lambda s^2, \quad u(\xi) = A_1\left[p + \tan(\phi(\xi)/2)\right] + B_1\left[p + \tan(\phi(\xi)/2)\right]^{-1}.$$

By using of (4.30) and Family 1, we can write

$$u_{34}(\xi) = \frac{\sqrt{2}}{2}B_1\tan\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi + C)\right) + \frac{\sqrt{2}}{2}B_1\cot\left(\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}B_1(\xi + C)\right), \tag{4.31}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda st^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{(\mu B_1^2 - \lambda s^2)t^\alpha}{\Gamma(1+\alpha)}$.

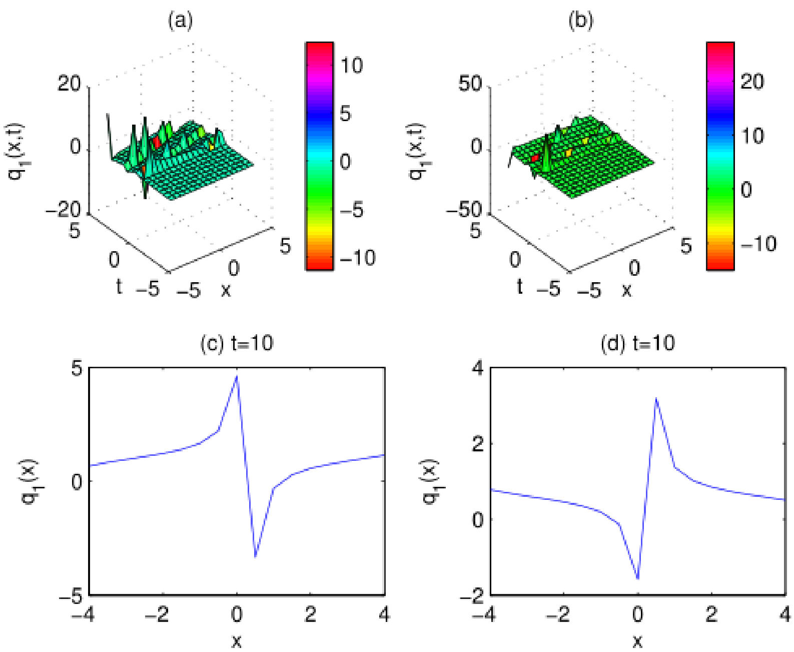

FIGURE 1. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 2, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

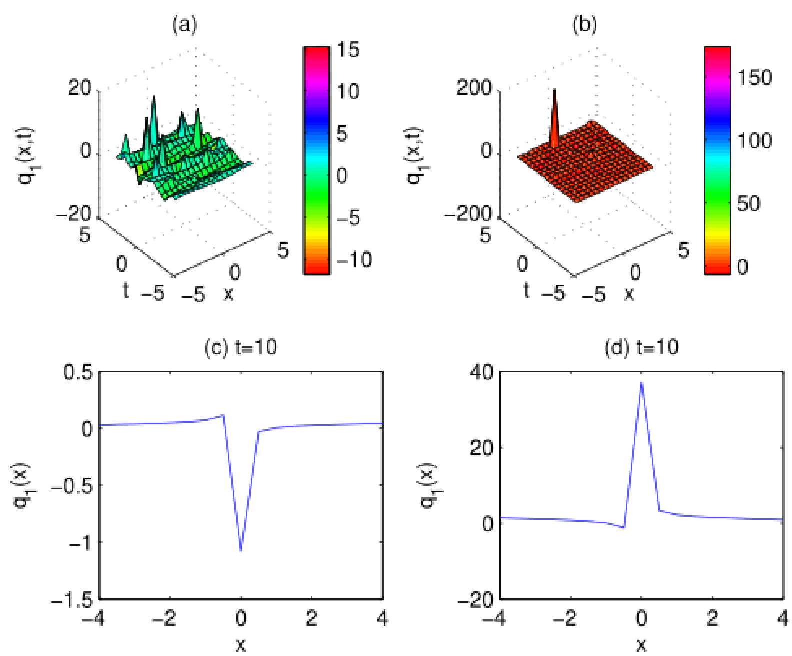

FIGURE 2. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 2, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

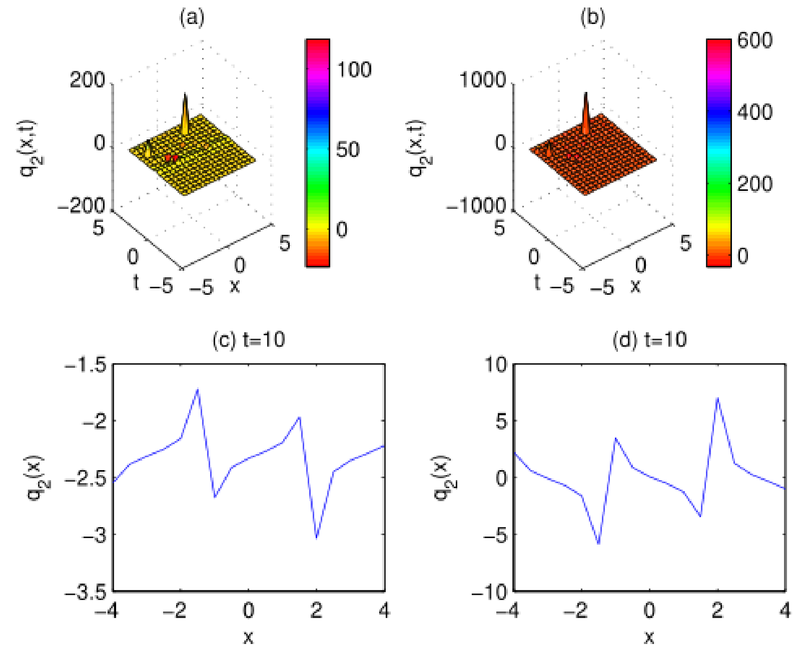

FIGURE 3. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 3, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

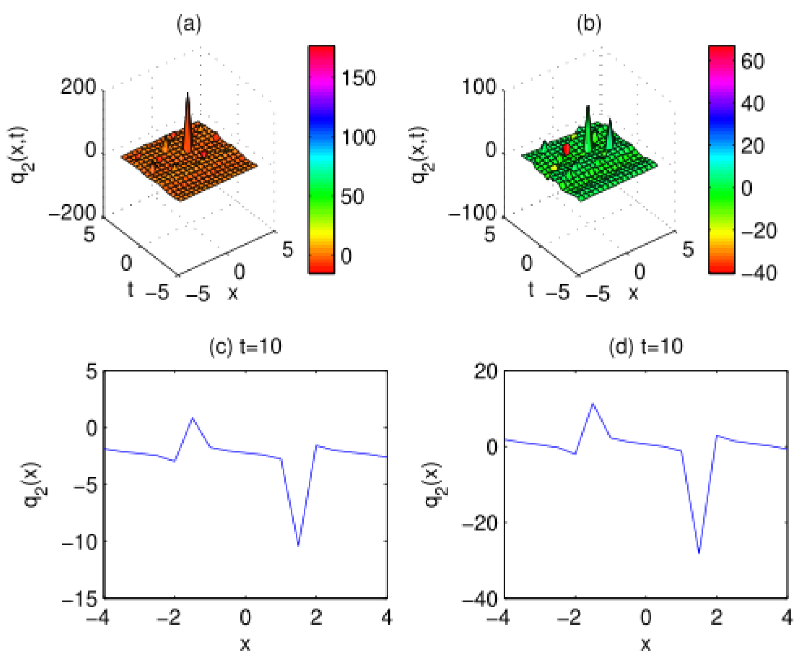

FIGURE 4. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 2, b = 3, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

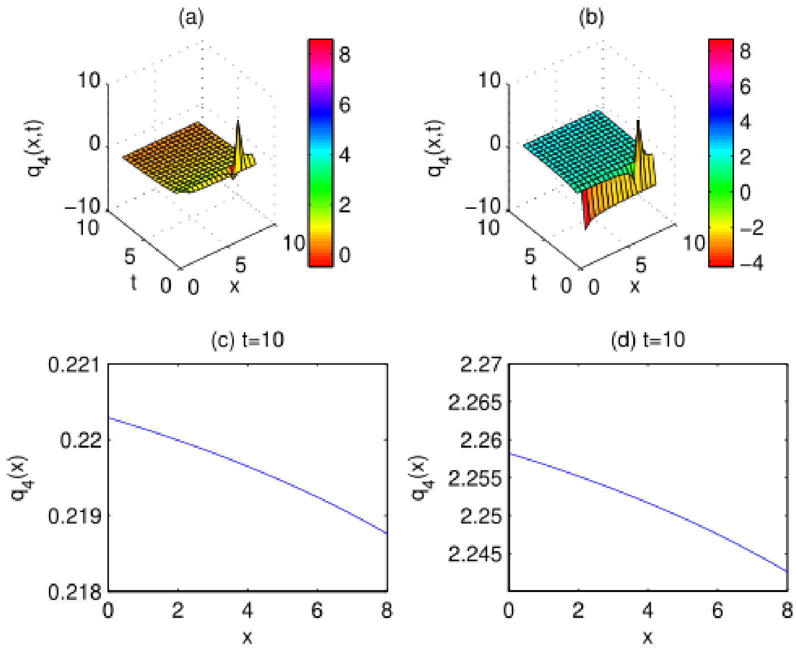

FIGURE 5. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 4, c = 5, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

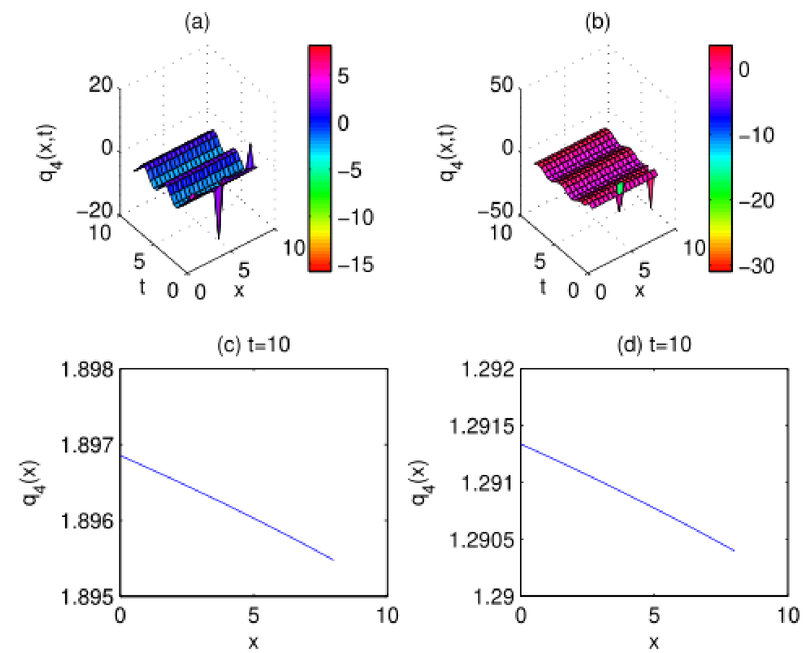

FIGURE 6. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 4, c = 5, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

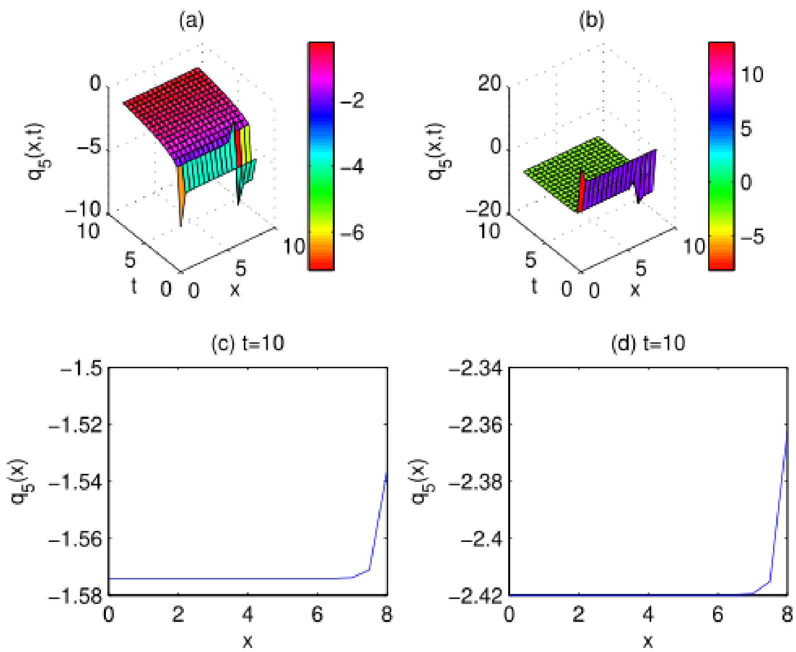

FIGURE 7. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 5, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.5, \beta = 0.1$.

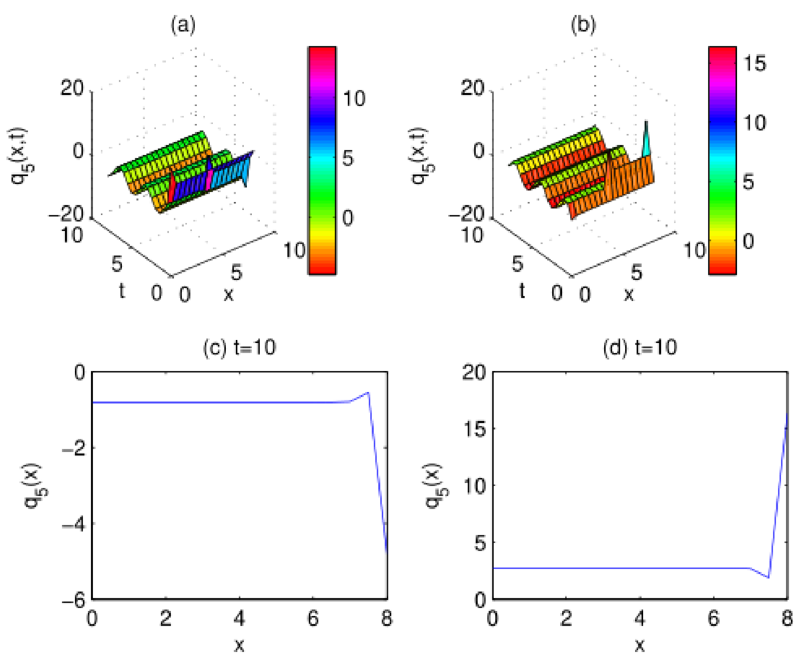

FIGURE 8. Panels ((a), (c)) and ((b), (d)) show the real and imaginary values respectively of (4.11), by considering the values $a = 3, b = 5, c = 3, p = 1, \lambda = 2, \mu = 3, s = 2, \alpha = 0.99, \beta = 0.99$.

Remark 4.1. In Figures 1-8, we plot two dimensional and three dimensional graphics of absolute values of (4.11), which denote the dynamics of solutions with appropriate parametric selections. We plot two and three dimensional graphics of Figs 1-4 when $-4 < x < 4, -4 < t < 4$. Moreover, we plot two and three dimensional graphics of Figs 5-8 when $0 < x < 8, 0 < t < 8$. To the best of our knowledge, these optical soliton solutions have not been submitted to literature in advance. Figures 1 and 2 show periodic wave solutions, Figure 3 and 4 present soliton wave solutions. Also, Figures 5 and 6 show rational wave solutions. Moreover, Figures 7 and 8 present exponential wave solutions. We test our results based on different α and β.

4.2. **The Case $\mathbf{n} \geq 2$ case for the fractional BME.** Now, we close this work by assuming $n \geq 2$ in (3.6), therefore, we have

$$-\mathrm{n}(\mathrm{r}+\mathrm{n}\lambda \mathrm{s}^2)\mathrm{u}^2(\xi)+\mathrm{n}(\mathrm{n}-1)\lambda(\mathrm{u}'(\xi))^2+\mathrm{n}\lambda\mathrm{u}(\xi)\mathrm{u}''(\xi)+\mu\mathrm{F}(\mathrm{u}^2(\xi))\mathrm{u}^2(\xi)=0. \tag{4.32}$$

In the presence of perturbation terms, the fractional BME with Kerr law nonlinearity $(\mathrm{F(w)} = \mathrm{w})$ is given by

$$-\mathrm{n}(\mathrm{r}+\mathrm{n}\lambda \mathrm{s}^2)\mathrm{u}^2(\xi)+\mathrm{n}(\mathrm{n}-1)\lambda(\mathrm{u}'(\xi))^2+\mathrm{n}\lambda\mathrm{u}(\xi)\mathrm{u}''(\xi)+\mu\mathrm{u}^4(\xi)=0. \tag{4.33}$$

In order to determine value of m, we balance the linear term of the highest order uu'' with the highest order nonlinear term u^4 in Eq. (4.33) we get

$$u(\xi) = A_m\left(\tan(\phi(\xi)/2)\right)^m + ..., \tag{4.34}$$

$$u^4(\xi) = A_m^4\left(\tan(\phi(\xi)/2)\right)^{4m} + ..., \tag{4.35}$$

$$\frac{du(\xi)}{d\xi} = \frac{m(c-b)}{2}A_m\left(\tan(\phi(\xi)/2)\right)^{m+1} + ..., \tag{4.36}$$

$$\frac{d^2u(\xi)}{d\xi^2} = \frac{m(m+1)(c-b)^2}{2}A_m\left(\tan(\phi(\xi)/2)\right)^{m+2} + ..., \tag{4.37}$$

$$u\frac{d^2u(\xi)}{d\xi^2} = \frac{m(m+1)(c-b)^2}{2}A_m^2\left(\tan(\phi(\xi)/2)\right)^{2m+2} + ... \tag{4.38}$$

By considering the homogeneous balance principle between the highest order derivatives uu'' and nonlinear terms u^4, we obtain $2m + 2 = 4m$, then $m = 1$. For simplicity we set $p = 0$ in (2.3). Then the trail solution is

$$u(\xi) = \sum_{k=-1}^{1} A_k(p + \tan^k(\phi/2))^k, \tag{4.39}$$

Substituting (4.39) and (2.4) into Eq. (4.33) and collecting all terms with the same order of $tan\,(\Phi(\xi)/2)$ together, and setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for a, b, c, k, w, A_0, A_1, and B_1 as follows:

Coefficients of $Y = \tan(\phi(\xi)/2)$

$$\begin{cases}
Y^0: & B_1^2(4\mu B_1^2+2n\lambda cb+2n^2\lambda cb+n^2\lambda b^2+n\lambda c^2+n^2\lambda c^2+n\lambda b^2)=0,\\
Y^1: & 2B_1(8A_0\mu B_1^2+2B_1n^2\lambda ab+B_1n\lambda ab+2B_1n^2\lambda ac+B_1n\lambda ac+n\lambda A_0b^2+n\lambda A_0c^2+2n\lambda A_0cb)=0,\\
Y^2: & 2B_1(8\mu A_1B_1^2-B_1n^2\lambda b^2+2B_1n^2\lambda a^2+12B_1\mu A_0^2+B_1n^2\lambda c^2-2B_1n^2\lambda s^2-2B_1nr+4n\lambda cbA_1\\
& +3n\lambda A_0ba-n^2\lambda b^2A_1+2n\lambda c^2A_1-n^2\lambda c^2A_1+2n\lambda b^2A_1-2n^2\lambda cbA_1+3n\lambda A_0ca)=0,\\
Y^3: & 2B_1(-2B_1n^2\lambda ab-B_1n\lambda ac+B_1n\lambda ab+2B_1n^2\lambda ac+24B_1\mu A_0A_1+8\mu A_0^3-4n^2\lambda abA_1-4nrA_0\\
& +2n\lambda A_0a^2-n\lambda A_0b^2-4n^2\lambda s^2A_0+n\lambda A_0c^2+8n\lambda abA_1-4n^2\lambda acA_1+8n\lambda caA_1)=0,\\
Y^4: & -n\lambda(8na^2A_1B_1-16a^2A_1B_1+2A_0baB_1-2A_0baA_1-2A_0caB_1-2A_0caA_1-nb^2B_1^2-4nb^2A_1B_1\\
& +b^2B_1^2+b^2A_1^2+8b^2A_1B_1-nb^2A_1^2-2ncbA_1^2-2cbB_1^2+2ncbB_1^2+2cbA_1^2-8c^2A_1B_1+4nc^2A_1B_1-nc^2B_1^2\\
& -nc^2A_1^2+c^2B_1^2+c^2A_1^2)-4n^2\lambda s^2A_0^2+4\mu A_0^4-4nrA_0^2-8nrA_1B_1+48\mu A_0^2A_1B_1-8n^2\lambda s^2A_1B_1+24\mu A_1^2B_1^2=0,\\
Y^5: & 2A_1(2n^2\lambda abA_1-n\lambda caA_1-n\lambda abA_1+2n^2\lambda acA_1+24A_1\mu A_0B_1+8\mu A_0^3-4nrA_0+n\lambda A_0c^2\\
& +2n\lambda A_0a^2-8n\lambda abB_1-n\lambda A_0b^2-4n^2\lambda s^2A_0+4n^2\lambda abB_1+8n\lambda caB_1-4n^2\lambda acB_1)=0,\\
Y^6: & 2A_1(8\mu B_1A_1^2-2nrA_1+n^2\lambda c^2A_1+2A_1n^2\lambda a^2-n^2\lambda b^2A_1-2n^2\lambda s^2A_1+12\mu A_0^2A_1+2n^2\lambda cbB_1+3n\lambda A_0ca\\
& -n^2\lambda B_1b^2+2n\lambda B_1c^2+2n\lambda B_1b^2-n^2\lambda B_1c^2-4n\lambda cbB_1-3n\lambda A_0ba)=0,\\
Y^7: & -2A_1(-8\mu A_0A_1^2-2n^2\lambda acA_1-n\lambda caA_1+2n^2\lambda abA_1+n\lambda abA_1-n\lambda A_0c^2-n\lambda A_0b^2+2n\lambda A_0cb)=0,\\
Y^8: & A_1^2(4\mu A_1^2+b^2\lambda n+c^2\lambda n+n^2\lambda c^2+n^2\lambda b^2-2n\lambda cb-2n^2\lambda cb)=0.
\end{cases}$$

(4.40)

Solving the set of algebraic equations using Maple, we get the following results:

Case I

$$a = 0, \quad b = \sqrt{-\frac{\mu}{n(n+1)\lambda}}B_1, \quad c = \sqrt{-\frac{\mu}{n(n+1)\lambda}}B_1, \quad p = 0, \quad A_0 = 0, \quad A_1 = 0, \tag{4.41}$$

$$B_1 = B_1, \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = B_1 \cot(\phi(\xi)/2).$$

By using of (4.41) and Family 16, we can write as

$$u_1(\xi) = \frac{B_1}{c(\xi + C)}, \tag{4.42}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^\alpha}{\Gamma(1+\alpha)}$.

Case II

$$a = 0, \quad b = -\sqrt{-\frac{\mu}{n(n+1)\lambda}}A_1, \quad c = \sqrt{-\frac{\mu}{n(n+1)\lambda}}A_1, \quad p = 0, \quad A_0 = 0, \quad A_1 = A_1, \tag{4.43}$$

$$B_1 = 0, \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = A_1 \tan(\phi(\xi)/2).$$

By using of (4.43) and Family 17, we can write as

$$u_2(\xi) = -\frac{A_1}{c(\xi + C)}, \tag{4.44}$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^\alpha}{\Gamma(1+\alpha)}$.

Case III

$$a = 0, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = 0, \quad A_1 = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b - c), \tag{4.45}$$

$$B_1 = \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b + c), \quad s = s, \quad r = n\lambda(b^2 - c^2 - s^2), \quad u(\xi) = A_1 \tan(\phi(\xi)/2) + B_1 \cot(\phi(\xi)/2).$$

By using of (4.45) and Families 5, 6, 11, 16, we can write as

$$u_3(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}\sqrt{b^2 - c^2}\left\{\tanh\left(\frac{\sqrt{b^2 - c^2}}{2}\bar{\xi}\right) - \coth\left(\frac{\sqrt{b^2 - c^2}}{2}\bar{\xi}\right)\right\}, \tag{4.46}$$

where $\bar{\xi} = \xi + C$, $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{n\lambda(b^2 - c^2 - s^2)t^\alpha}{\Gamma(1+\alpha)}$.

$$u_4(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}b\left\{\tan\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}} - 1}{e^{2b\bar{\xi}} + 1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}} + 1}\right]\right) - \cot\left(\frac{1}{2}\arctan\left[\frac{e^{2b\bar{\xi}} - 1}{e^{2b\bar{\xi}} + 1}, \frac{2e^{b\bar{\xi}}}{e^{2b\bar{\xi}} + 1}\right]\right)\right\},$$

$$u_5(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{4\mu}}b\left\{\frac{be^{b(\xi+C)} - 1}{be^{b(\xi+C)} + 1} - \frac{be^{b(\xi+C)} + 1}{be^{b(\xi+C)} - 1}\right\},$$

where $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^\beta}{\Gamma(1+\beta)} + \frac{n\lambda(b^2 - s^2)t^\alpha}{\Gamma(1+\alpha)}$.

$$u_6(\xi) = -\sqrt{-\frac{n(n+1)\lambda}{\mu}}\frac{1}{(\xi + C)}, \quad q(x,t) = u(\xi)e^{i\theta}, \quad \xi = x - \frac{2\lambda nst^\alpha}{\Gamma(1+\alpha)} \quad \theta = \frac{st^\beta}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^\alpha}{\Gamma(1+\alpha)},$$

Case IV

$$a = -\sqrt{c^2 - b^2}, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = \sqrt{-\frac{(b^2 - c^2)n(n+1)\lambda}{4\mu}}, \quad A_1 = \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b - c), \tag{4.47}$$

$$B_1 = 0, \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = A_0 + A_1 \tan(\phi(\xi)/2).$$

By using of (4.47) and Families 8, we can write as

$$u_7(\xi) = \sqrt{-\frac{(b^2 - c^2)n(n+1)\lambda}{4\mu}} + \sqrt{-\frac{n(n+1)\lambda}{4\mu}} \frac{a\bar{\xi} + 2}{\bar{\xi}}, \tag{4.48}$$

where $\bar{\xi} = \xi + C$, $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^{\alpha}}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^{\beta}}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^{\alpha}}{\Gamma(1+\alpha)}$.

Case V

$$a = \sqrt{c^2 - b^2}, \quad b = b, \quad c = c, \quad p = 0, \quad A_0 = \sqrt{-\frac{(b^2 - c^2)n(n+1)\lambda}{4\mu}}, \quad A_1 = 0, \tag{4.49}$$

$$B_1 = \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b + c), \quad s = s, \quad r = -n\lambda s^2, \quad u(\xi) = A_0 + B_1 \cot(\phi(\xi)/2).$$

By using of (4.49) and Families 8, we can write as

$$u_8(\xi) = \sqrt{-\frac{(b^2 - c^2)n(n+1)\lambda}{4\mu}} + \sqrt{-\frac{n(n+1)\lambda}{4\mu}}(b^2 - c^2)\frac{\bar{\xi}}{a\bar{\xi} + 2}, \tag{4.50}$$

where $\bar{\xi} = \xi + C$, $q(x,t) = u(\xi)e^{i\theta}$, $\xi = x - \frac{2\lambda nst^{\alpha}}{\Gamma(1+\alpha)}$, and $\theta = \frac{st^{\beta}}{\Gamma(1+\beta)} - \frac{n\lambda s^2 t^{\alpha}}{\Gamma(1+\alpha)}$.

5. Conclusion

In this paper, we presented the improved $\tan(\Phi(\xi)/2)$-expansion method for solving the fractional Biswas-Milovic (FBM) equation. We obtained abundant results for FBME. The exact particular solutions contain four types: the hyperbolic function, the trigonometric function, the exponential, and the rational solution. Abundant exact travelling wave solutions including solitons, kink, periodic, and rational solutions are attained. It is worth mentioning that some of newly obtained solutions are identical to already published results. It has been shown that the applied method is effective and more wide-ranging than the Exp-function method and sine-cosine method because it gives many new solutions. Therefore, this method can be applied to study many other nonlinear partial differential equations which frequently arise in engineering, mathematical physics, and nonlinear optics.

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