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Hyers-Ulam and exponential stabilities of autonomous and non-autonomous difference equations

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Abstract

In this manuscript, we studied the Hyers-Ulam and exponential stabilities of autonomous and non-autonomous difference equations of first and second order. Ultimately, we provide some examples to support our results.

Keywords. Hyer-Ulam, Exponential stability, Difference equations.

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1. Introduction

Differential equations have been used to analyze a variety of biological systems and models for a very long time in mathematics. Differential equations have several uses in a variety of disciplines, including engineering, economics, statistics, and the natural sciences [9, 16–18]. Even while differential equations are quite useful, some real-world issues call for discrete data sampling, which necessitates the usage of difference equations [6, 8]. Difference equations have become crucial tools in the study of biological systems as well as in the sciences, technology and mathematical physics. They offer a structure for precisely describing phenomena, such as wind flow dynamics and monographs, which helps in problem-solving in these fields.

Stability is one of the key qualitative characteristics of different systems that has a vital impact. System performance must be ensured by stability assessment. Due to its applicability in a variety of situations, Hyers-Ulam stability has attracted a lot of attention from researchers. The concept of Hyers-Ulam stability emerged almost eighty years ago in 1940 when Ulam [30] put a question before mathematicians at a conference about the existence of a group homomorphism that is near to an approximate group homomorphism on groups. After a year in 1941, Hyers [11] addressed Ulam's puzzles by considering the group as a Banach space and provided solutions. Consequently, the stability was named Hyers-Ulam stability. Donald H. Hyers made significant contributions in this field by providing a partially positive response to Ulam's query in the case of additive mappings within Banach spaces. Since then, many works have been published regarding various extensions of Ulam's problem and Hyers' theorem. In 1978, Rassias [21] provided an overview of this stability and applied the concept to the Cauchy problem. Furthermore, Obloza [20] use this concept in the field of differential equations, while Jung [13] and Khan et al. [14] discussed the idea for difference equations. Z. Gao et al. [10] and S. O. Shah et al. [22] reported the same stability results on a delayed first-order nonlinear dynamic system. Recent studies have focused on generalized Hyers-Ulam stability. A. R. Aruldass [1]. Additionally, investigations were made on the Hyers-Ulam stability of differential equations of order two utilizing the Mahgoub type. For other models and their solution we refer to [3, 23]. Hyers-Ulam stability is a mathematical concept related to functional equations. Hyers-Ulam stability is a mathematical concept that deals with the stability

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of functional equations. More precisely, it concerns the question of whether small perturbations of the inputs of a functional equation lead to correspondingly small perturbations of the output. Additionally, Ulam type stabilities several different mathematical models have been investigated [24–29] and some interesting results have been obtained in these sources.

In this paper, we describe the exponential stability and Hyers-Ulam stability of autonomous and non-autonomous difference systems of the type.

$$\begin{cases} y_{n+1} = (1+r)y_n, \quad n \geq 0, \\ y_0 = \theta, \end{cases} \tag{1.1}$$

$$\begin{cases} y_{n+2} = (1+r)y_n, \quad n \geq 0, \\ y_0 = a, \\ y_1 = b, \end{cases} \tag{1.2}$$

$$\begin{cases} y_{n+1} = r_n y_n, \quad n \geq 0, \\ y_0 = \theta, \end{cases} \tag{1.3}$$

and

$$\begin{cases} y_{n+2} = r_n y_n, \quad n \geq 0, \\ y_0 = \alpha, \\ y_1 = \beta, \end{cases} \tag{1.4}$$

where $r, r_n \in \mathbb{R}$, $y_n \in \mathbb{C}(\mathbb{Z}_+, \mathbf{X})$ where $\mathbb{C}(\mathbb{Z}_+, \mathbf{X})$ is the space of convergent sequences equiped with norm supremum, $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\mathbf{X} = \mathbb{R}^m$.

For other recent work related to the Hyers-Ulam stability of different type of difference equations we refer to [2, 5, 12, 15, 19].

2. Preliminaries

Throughout the paper, we will use the following notations: the vector norm will be denoted by $\|\cdot\|$, the n-dimensional Euclidean space will be represented by the symbol $\mathbf{X} = \mathbb{R}^m$. The collection of real, integer, and non-negative integer numbers are represented by the symbols $\mathbb{R}, \mathbb{Z}$, and $\mathbb{Z}_+$, respectively.

Definition 2.1. We recall that a family $\mathcal{E} = \{E(n,k) : n \geq k \geq 0\}$ of bounded linear operators acting on a Banach space $\mathbf{X}$ is called a discrete evolution family if it satisfies the following two conditions:

(i) $E(n,n) = I$ for all $n \geq 0$,
(ii) $E(n,s)E(s,k) = E(n,k)$ for all $n \geq s \geq k \geq 0$.

In the solutions of non-autonomous systems, we can take the evolution family as given below:

$$E(n,k) := \begin{cases} I, & \text{if all } n = k; \\ r_{n-1}r_{n-2}r_{n-3}...r_k, & \text{for all } n \geq k \geq 0. \end{cases}$$

Definition 2.2. The systems (1.1) to (1.4) are said to be uniformly exponentially stable if there are two positive real numbers ξ and μ, such that $\|y_n\| \leq \xi e^{-\mu n}, \ \forall n \geq 0$.

Definition 2.3. A sequence Ψ_n is referred to as an ϵ-approximate solution of (1.1), (1.2), (1.3), and (1.4) if the following inequalities hold

$$\|\Psi_{n+1} - (1+r)\Psi_n\| \leqslant \epsilon, \quad n \geq 0, \tag{2.1}$$

$$\|\Psi_{n+2} - (1+r)\Psi_n\| \leqslant \epsilon, \quad n \geq 0, \tag{2.2}$$

$$\|\Psi_{n+1} - r_n\Psi_n\| \leqslant \epsilon, \quad n \geq 0, \tag{2.3}$$

$$\|\Psi_{n+2} - r_n\Psi_n\| \leqslant \epsilon, \quad n \geq 0. \tag{2.4}$$

Remark 2.4. From Definition 2.3, we have that $y_n \in \mathbb{C}(\mathbb{Z}_+, \mathbf{X})$ will satisfy (2.1), (2.2), (2.3), and (2.4) if and only if there is a sequence $f \in \mathbb{C}(\mathbb{Z}_+, \mathbf{X})$ with $\|f_n\| \leqslant \epsilon$ such that

$$\begin{cases} y_{n+1} = (1+r)y_n + f_n, & n \geq 0, \\ y_0 = \theta, \end{cases}$$

$$\begin{cases} y_{n+2} = (1+r)y_n + f_n, & n \geq 0, \\ y_0 = a \\ y_1 = b, \end{cases}$$

$$\begin{cases} y_{n+1} = r_n y_n + f_n, & n \geq 0, \\ y_0 = \theta, \end{cases}$$

$$\begin{cases} y_{n+2} = r_n y_n + f_n, & n \geq 0, \\ y_0 = \alpha, \\ y_1 = \beta. \end{cases}$$

Lemma 2.5. *The systems (1.1), (1.2), (1.3), and (1.4) have the solutions:*

$$y_n = (1+r)^n \theta,$$

$$y_n = \begin{cases} (1+r)^{\frac{n}{2}} a, & \text{when } n \text{ is even}, \\ (1+r)^{\frac{n-1}{2}} b, & \text{when } n \text{ is odd}, \end{cases}$$

$$y_n = E(n,0)\theta,$$

and

$$y_n = \begin{cases} \prod_{\lambda=1}^{\frac{n}{2}} r_{n-2\lambda}\alpha, & \text{when } n \text{ is even}, \\ \prod_{\lambda=1}^{\frac{n-1}{2}} r_{n-2\lambda}\beta, & \text{when } n \text{ is odd}. \end{cases}$$

respectively.

Lemma 2.6. *The systems (1.1), (1.2), (1.3), and (1.4) have the approximate solutions:*

$$\varphi_n = (1+r)^n \gamma + \sum_{\lambda=0}^{n-1} (1+r)^{n-1-k} f_\lambda,$$

$$\psi_n = \begin{cases} (1+r)^{\frac{n}{2}} a + \sum_{\lambda=0}^{\frac{n}{2}-1} (1+r)^{\frac{n}{2}-1-\lambda} f_{2\lambda}, & n = 2,4,6,8,... \\ (1+r)^{\frac{n-1}{2}} b + \sum_{\lambda=0}^{\frac{n-1}{2}-1} (1+r)^{\frac{n-1}{2}-1-\lambda} f_{2\lambda+1}, & n = 1,3,5,7,... \end{cases}$$

$$\phi_n = E(n,0)\theta + \sum_{\lambda=1}^{n} E(n,\lambda) f_{\lambda-1},$$

and

$$\Psi_n = \begin{cases} \prod_{\lambda=1}^{\frac{n}{2}} (r_{n-2\lambda})\alpha + \sum_{j=1}^{\frac{n}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}, & n = 2,4,6,8,..., \\ \prod_{\lambda=1}^{\frac{n-1}{2}} (r_{n-2\lambda})\beta + \sum_{j=1}^{\frac{n-1}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}, & n = 1,3,5,7,..., \end{cases}$$

respectively.

The solutions in Lemmas 2.5 and 2.6 can easily be obtained by putting the values of n.

Definition 2.7. *The systems (1.1), (1.2), (1.3), and (1.4) are said to be Hyers-Ulam stable if there exists a positive real numbers* $\mathbf{L}$ *such that for every approximate solutions* ψ_n *of the system (1.1), (1.2), (1.3), and (1.4) there are solutions* y_n *of (1.1), (1.2), (1.3), and (1.4) such that*

$$\|y_n - \psi_n\| \leq \mathbf{L}\epsilon, \quad n \in I.$$

Remark 2.8. [7] Every semigroup is an evolution, but the converse is not true in general.

Remark 2.9. [7] An evolution family is a semigroup if it is periodic of period one.

From Remarks 2.8 and 2.9, the relation between autonomous and non-autonomous is very clear that every autonomous may be considered as non-autonomous, but for the reverse it is necessary that the time-dependent matrix must be periodic with period 1.

3. Hyers-ulam stability of autonomous systems

In this section, we study the Hyers-Ulam stability of autonomous difference systems of first and second order. For both the results, we need the following assumption:

$$\Lambda_1 : |1+r| < 1.$$

Theorem 3.1. *If Λ_1 holds, then the system (1.1) is Hyer-Ulam stable.*

Proof. By Lemmas 2.5 and 2.6 the system (1.1) has the exact and approximate solutions:

$$y_n = (1+r)^n \gamma,$$

and

$$\varphi_n = (1+r)^n \gamma + \sum_{\lambda=0}^{n-1} (1+r)^{n-1-k} f_\lambda,$$

respectively.

Now consider

$$\begin{aligned}
\|y_n - \varphi_n\| &= \|(1+r)^n\gamma - (1+r)^n\gamma - \sum_{\lambda=0}^{n-1}(1+r)^{n-1-\lambda} f_\lambda\| \\
&= \|\sum_{\lambda=0}^{n-1}(1+r)^{n-1-\lambda} f_\lambda\| \leqslant \varepsilon \sum_{\lambda=0}^{n-1}(1+r)^{n-1-\lambda} \\
&= \varepsilon(a^{n-1} + a^{n-2} + ... + a^0) = \varepsilon(1 + a^1 + a^2 + ... + a^{n-1}) \\
&\leqslant \varepsilon(1 + a^1 + a^2 + ...) = \frac{\varepsilon}{1-a} = \frac{\varepsilon}{1-(1+r)} = \frac{\varepsilon}{-r} = L\varepsilon,
\end{aligned}$$

where $L = \frac{1}{-r}$, hence the system is Hyers-Ulam stable. □

Theorem 3.2. *If Λ_1 holds, then the system (1.2) is Hyer-Ulam stable.*

Proof. By Lemmas 2.5 and 2.6, the system (1.2) has the exact and approximate solutions

$$y_n = \begin{cases} (1+r)^{\frac{n}{2}} a, & n = 2,4,6,8,..., \\ (1+r)^{\frac{n-1}{2}} b, & n = 1,3,5,7,..., \end{cases}$$

and

$$\psi_n = \begin{cases} (1+r)^{\frac{n}{2}} a + \sum_{\lambda=0}^{\frac{n}{2}-1} (1+r)^{\frac{n}{2}-1-\lambda} f_{2\lambda}, & n = 2,4,6,8,..., \\ (1+r)^{\frac{n-1}{2}} b + \sum_{\lambda=0}^{\frac{n-1}{2}-1} (1+r)^{\frac{n-1}{2}-1-\lambda} f_{2\lambda+1}, & n = 1,3,5,7,..., \end{cases}$$

Now if n is even, we have

$$\begin{aligned}
\|y_n - \psi_n\| &= \|(1+r)^{\frac{n}{2}} a - (1+r)^{\frac{n}{2}} a - \sum_{\lambda=0}^{\frac{n}{2}-1}(1+r)^{\frac{n}{2}-1-\lambda} f_{2\lambda}\| \\
&= \|\sum_{\lambda=0}^{\frac{n}{2}-1}(1+r)^{\frac{n}{2}-1-\lambda} f_{2\lambda}\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \varepsilon \sum_{\lambda=0}^{\frac{n}{2}-1} |1+r|^{\frac{n}{2}-1-\lambda} \\
&\leqslant \varepsilon \sum_{\lambda=0}^{\infty} |1+r|^{\lambda} = \varepsilon(1+c^1+c^2+c^3+...) \\
&= \varepsilon \frac{c}{1-c} \\
&= \varepsilon \frac{1+r}{1-(1+r)} \\
&= \varepsilon \frac{1+r}{-r} \\
&= L\varepsilon. \qquad (3.1)
\end{aligned}
$$

Now if n is odd, we have

$$
\begin{aligned}
\|y_n-\psi_n\| &= \|(1+r)^{\frac{n-1}{2}}b-(1+r)^{\frac{n-1}{2}}b-\sum_{\lambda=0}^{\frac{n-1}{2}-1}(1+r)^{\frac{n-1}{2}-1-\lambda}f_{2\lambda+1}\| \\
&= \|\sum_{\lambda=0}^{\frac{n-1}{2}-1}(1+r)^{\frac{n-1}{2}-1-\lambda}f_{2\lambda+1}\| \\
&\leqslant \varepsilon \sum_{\lambda=0}^{\frac{n-1}{2}-1}(1+r)^{\frac{n-1}{2}-1-\lambda} \\
&\leqslant \varepsilon \sum_{\lambda=0}^{\infty} |1+r|^{\lambda} \\
&= \varepsilon(1+c^1+c^2+c^3+...) \\
&= \varepsilon \frac{c}{1-c} \\
&= \varepsilon \frac{1+r}{1-(1+r)} \\
&= \varepsilon \frac{1+r}{-r} \\
&= L\varepsilon, \qquad (3.2)
\end{aligned}
$$

where $L=\frac{1+r}{-r}$, hence the system is Hyers-Ulam stable. □

4. Hyers-ulam stability of non-autonomous systems

In this section, we study the Hyers-Ulam stability of non-autonomous difference systems of first and second order. For the next two results, we need the following assumptions:

$$\Lambda_3 : |r(n,\lambda+1)| \leq Me^{-\gamma(n-\lambda-1)},$$

$$\Lambda_4 : |\prod_{\lambda=1}^{j}(r_{n-2\lambda})| < Me^{-\mu(j-1)}.$$

Theorem 4.1. *If Λ_3 holds, then the system (1.3) is Hyer-Ulam stable.*

Proof. By Lemmas 2.5 and 2.6 the system (1.3) has the exact and approximate solutions

$$y_n = E(n,0)\theta,$$

and

$$\phi_n = E(n,0)\theta + \sum_{\lambda=1}^{n} E(n,\lambda) f_{\lambda-1},$$

where $E(n,k)$ is the evolution family. Now consider

$$\begin{aligned}
\|y_n - \phi_n\| &= \|r(n,0)\theta - r(n,0)\theta + \sum_{\lambda=0}^{n-2} r(n,\lambda+1) f_\lambda + f_{n-1}\| \\
&= \| - \sum_{\lambda=0}^{n-2} r(n,\lambda+1) f_\lambda + f_{n-1}\| = \| \sum_{\lambda=0}^{n-2} r(n,\lambda+1) f_\lambda + f_{n-1}\| \\
&\leqslant \epsilon M \sum_{\lambda=0}^{n-2} e^{-\gamma(n-\lambda-1)} \\
&= \epsilon M (e^{-\gamma(n-1)} + e^{-\gamma(n-2)} + e^{-\gamma(n-3)} + .. + 1) \\
&\leqslant \epsilon M (1 + e^{-\gamma} + e^{-2\gamma} + e^{-3\gamma} + e^{-4\gamma} + ...) \\
&= \epsilon M \frac{1}{1-e^{-\gamma}} = \epsilon M \frac{1}{1-\frac{1}{e^\gamma}} \\
&= \epsilon M \frac{e^\gamma}{e^\gamma - 1} = \epsilon L, \qquad (4.1)
\end{aligned}$$

where $L = M\frac{e^\gamma}{e^\gamma-1}$, hence the system (1.3) is Hyers-Ulam stable. □

Theorem 4.2. *If Λ_4 holds, then the system (1.4) is Hyers-Ulam stable.*

Proof. By Lemmas 2.5 and 2.6 the system (1.4) has the exact and approximate solutions

$$y_n = \begin{cases} \prod_{\lambda=1}^{\frac{n}{2}} r_{n-2\lambda}\alpha, & n = 2,4,6,8,..., \\ \prod_{\lambda=1}^{\frac{n-1}{2}} r_{n-2\lambda}\beta, & n = 1,3,5,7,..., \end{cases}$$

and

$$\Psi_n = \begin{cases} \prod_{\lambda=1}^{\frac{n}{2}} (r_{n-2\lambda})\alpha + \sum_{j=1}^{\frac{n}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}, & n = 2,4,6,8,..., \\ \prod_{\lambda=1}^{\frac{n-1}{2}} (r_{n-2\lambda})\beta + \sum_{j=1}^{\frac{n-1}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}, & n = 1,3,5,7,.... \end{cases}$$

Now if n is even, we have

$$\begin{aligned}
\|y_n - \Psi_n\| &= \| \prod_{\lambda=1}^{\frac{n}{2}} (r_{n-2\lambda})\alpha - \prod_{\lambda=1}^{\frac{n}{2}} (r_{n-2\lambda})\alpha - \sum_{j=1}^{\frac{n}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}\| \\
&= \| \sum_{j=1}^{\frac{n}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}\| \leqslant \epsilon \| \sum_{j=1}^{\frac{n}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda})\| \\
&\leqslant \epsilon M \sum_{j=1}^{\frac{n}{2}-1} e^{-\mu(j-1)} = \epsilon M (1 + e^{-\mu} + e^{-2\mu} + e^{-3\mu} + .. + e^{-(\frac{n-4}{2})\mu} \\
&\leqslant \epsilon M (1 + e^{-\mu} + e^{-2\mu} + e^{-3\mu} + ...) = \epsilon M \frac{1}{1-e^{-\mu}} = \epsilon M \frac{e^\mu}{e^\mu - 1} = \epsilon L. \qquad (4.2)
\end{aligned}$$

Now if n is odd, we have

$$\|y_n - \Psi_n\| = \| \prod_{\lambda=1}^{\frac{n-1}{2}} (r_{n-2\lambda})\beta - \prod_{\lambda=1}^{\frac{n-1}{2}} (r_{n-2\lambda})\beta - \sum_{j=1}^{\frac{n-1}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2}\|$$

$$= \| \sum_{j=1}^{\frac{n-1}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) f_{n-2\lambda-2} + f_{n-2} \| \leqslant \epsilon \| \sum_{j=1}^{\frac{n-1}{2}-1} \prod_{\lambda=1}^{j} (r_{n-2\lambda}) \|$$

$$\leqslant \epsilon M \sum_{j=1}^{\frac{n-1}{2}-1} e^{-\mu(j-1)} = \epsilon M (1 + e^{-\mu} + e^{-2\mu} + e^{-3\mu} + .. + e^{-(\frac{n-5}{2})\mu}$$

$$\leqslant \epsilon M (1 + e^{-\mu} + e^{-2\mu} + e^{-3\mu} + ...) = \epsilon M \frac{1}{1 - e^{-\mu}} = \epsilon M \frac{e^{\mu}}{e^{\mu} - 1} = \epsilon L, \tag{4.3}$$

where $L = M \frac{e^{\mu}}{e^{\mu}-1}$, hence the system is Hyers-Ulam stable. □

5. Uniform exponential stability

In this section, we study the uniform exponential stability of autonomous difference systems.

Theorem 5.1. *If Λ_1 holds, then the system (1.1) is uniformly exponentially stable.*

Proof. System (1.1) has the solution

$$y_n = (1 + r)^n \gamma.$$

Consider

$$\|y_n\| = \|a^n \gamma\| \leqslant |\gamma| \|a\|^n \leq \xi e^{-\mu n}, \tag{5.1}$$

where $a = 1 + r$ and $a^n = e^{\ln |a|^n} = e^{n \ln |a|} = e^{-\mu n}$, $0 < a < 1$, $\ln |a| = -\mu$, hence the system is uniformly exponentially stable. □

Theorem 5.2. *If Λ_1 holds, then the system (1.2) is uniformly exponentially stable.*

Proof. The system (1.2) has the solution:

$$y_n = \begin{cases} (1 + r)^{\frac{n}{2}} a, & n = 2, 4, 6, 8, ..., \\ (1 + r)^{\frac{n-1}{2}} b, & n = 1, 3, 5, 7, ... \end{cases}$$

Now if n is even, we have

$$\|y_n\| = \|(1 + r)^{\frac{n}{2}} a\| = \|\lambda^{\frac{n}{2}} a\| \leqslant |a| \|\lambda\|^{\frac{n}{2}} \leqslant |a| \|\lambda\|^n \leqslant \Psi e^{-\mu n}. \tag{5.2}$$

Now if n is odd, we have

$$\|y_n\| = \|(1 + r)^{\frac{n-1}{2}} b\| = \|\lambda^{\frac{n-1}{2}} b\| \leqslant |b| \|\lambda\|^{\frac{n-1}{2}} \leqslant |b| \|\lambda\|^n \leqslant \Psi e^{-\mu n}, \tag{5.3}$$

where $\lambda = 1 + r$ and $\lambda^n = e^{\ln |\lambda|^n} = e^{n \ln |\lambda|} = e^{-\mu n}$, $\quad 0 < \lambda < 1 \quad \ln |\lambda| = -\mu$, hence the system is UES. □

6. Examples

In this section, we will present some examples which will elaborate on the above results. For the autonomous and first order difference equation, we have the following example.

Example 6.1. Consider the autonomous difference equation:

$$\begin{cases} y_{n+1} = (1 + r) y_n, \quad r = -0.5, \quad n \geq 0, \\ y_0 = 0.3. \end{cases}$$

Then the following will be its solution:

$$y_n = (1 + r)^n \gamma = (0.5)^n (0.3).$$

For approximate solution, let $f_n = \varepsilon 0.6$ and consider the purterb system:

$$\begin{cases} \varphi_{n+1} = (1 + r) \varphi_n + \varepsilon 0.6, \quad r = -0.5, \quad n \geq 0, \\ \varphi_0 = 0.3. \end{cases}$$

Its solution is

$$\varphi_n = (0.5)^n(0.3) + \sum_{j=0}^{n-1}(0.5)^{n-1-j}(\varepsilon 0.6),$$

which is the approximate solution of the above system.

Now consider

$$\|y_n - \varphi_n\| = \|(0.5)^n(0.3) - (0.5)^n(0.3) - \sum_{j=0}^{n-1}(0.5)^{n-1-j}(\varepsilon 0.6)\|$$

$$\leq (\varepsilon 0.6)\sum_{j=0}^{n-1}(0.5)^{n-1-j}$$

$$\leq (\varepsilon 0.6)\sum_{j=0}^{\infty}(0.5)^{j} = \frac{\varepsilon 0.6}{0.5} = \frac{6}{5}\varepsilon.$$

Thus by Theorem 3.1 the above system is Hyer-Ulam stable.

Now for the uniform exponential stability, we have $\|y_n\| \leq 0.3e^{-\mu n}$, where $\mu = \ln 2$, since $e^{-\mu n} = (0.5)^n \Rightarrow \ln(e^{-\mu n}) = \ln(0.5)^n \Rightarrow -\mu n = n\ln(0.5) \Rightarrow -\mu = -\ln 2 \Rightarrow \mu = \ln 2$. Hence, by Theorem 5.1 the system is also unifromly exponentially stable.

For a second order autonomous system, we have the following example.

Example 6.2. Consider the second order difference equation:

$$\begin{cases} y_{n+2} = \frac{1}{2}y_n, \quad n \geq 0, \\ y_0 = a, \\ y_1 = b. \end{cases} \tag{6.1}$$

The solution is given as

$$y_n = \begin{cases} (\frac{1}{2})^{\frac{n}{2}}a, \ when \text{ n } is\ even, \\ (\frac{1}{2})^{\frac{n-1}{2}}b, \ when \text{ n } is\ odd. \end{cases}$$

Since the axiom λ_1 is satisfied so the system is Hyer-Ulam as well as uniformly exponentially stable.

The next example is for the case of a non-autonomous system.

Example 6.3. Consider the non-autonomous difference equation:

$$\begin{cases} y_{n+1} = \frac{1}{2^n}y_n, \quad n \geq 0, \\ y_0 = 0.3. \end{cases}$$

Then the following will be its solution:

$$y_n = E(n,0)\theta = \frac{1}{2^{n-1}}\frac{1}{2^{n-2}}\frac{1}{2^{n-3}}\cdots\frac{1}{2^0}\theta = \frac{1}{2^{\frac{n(n-1)}{2}}}\theta.$$

The approximate solution of the above system will be:

$$\phi_n = E(n,0)\theta + \sum_{\lambda=1}^{n}E(n,\lambda)f_{\lambda-1} = \frac{1}{2^{\frac{n(n-1)}{2}}}\theta + \sum_{\lambda=1}^{n}\frac{1}{2^{\frac{n(n-1)-\lambda(\lambda-1)}{2}}}f_{\lambda-1},$$

where $\|f\| \leq \epsilon$.

Consider

$$\|y_n - \phi_n\| = \|\sum_{\lambda=1}^{n}\frac{1}{2^{\frac{n(n-1)-\lambda(\lambda-1)}{2}}}f_{\lambda-1}\| \leq \epsilon\sum_{\lambda=1}^{n}\frac{1}{2^\lambda} \leq \epsilon\sum_{\lambda=1}^{\infty}\frac{1}{2^\lambda} = 1.\epsilon,$$

hence by Theorem 4.1, the system is Hyers-Ulam stable.

For unifrom exponential stability, we have

$$\|y_n\| = \|\frac{1}{2^{\frac{n(n-1)}{2}}}\theta\| \leq \|\theta\|\frac{1}{2^n},$$

which goes to zero exponentially hence by Theorem 5.1, the system is uniformly exponentially stable.

On the same way, we can easily construct an example for a non-autonomous second order difference system which we leave for the reader.

7. Cocnlusion

The study of difference equations is too important and has a lot of applications in different fields of science. The main purpose of this paper is to discuss the solutions, Hyer-Ulam and uniform exponential stabilities of autonomous and non-autonomous difference equations of first and second order. These models have not been discussed before and therefore it can be fruitful for other researchers working in the field of difference equations and stability theory. Simple conditions were provided for each system through which one can easily check the stability of the proposed models. Some examples are also given to ensure the credibility of the main results.

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