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Existence and uniqueness theorems for fractional differential equations with proportional delay

Prajakta Rajmane[1,*], Jayvant Patade[2,3], and Machchhindra Gophane[1]
[1]Department of Mathematics, Shivaji University, Kolhapur - 416004, India.
[2]Department of Mathematics, Jaysingpur College, Jaysingpur (Affiliated to Shivaji University, Kolhapur) - 416101, India.
[3]Department of Mathematics, Tuljaram Chaturchand College, Baramati (Affiliated to Savitribai Phule Pune University, Pune) - 413102, India.

Abstract

In this paper, we apply the successive approximation method (SAM) to solve nonlinear differential equations (DEs) with proportional delay. Utilizing SAM, we establish results on existence and uniqueness. Differential equations (DEs) with proportional delay represent a particular case of time-dependent delay differential equations (DDEs). We demonstrate that the equilibrium solution of time-dependent DDEs is asymptotically stable over finite time intervals. We obtained a series solution for the pantograph and Ambartsumian equations and proved their convergence. Furthermore, we prove that the zero solution of the pantograph and Ambartsumian equations is asymptotically stable. The outcomes of integer order obtained for DEs with proportional delay and time-dependent DDEs have been extended to the initial value problem (IVP) for fractional DDEs and a system of fractional DDEs involving the Caputo fractional derivative. Finally, we illustrate SAM's efficacy using particular non-linear DEs with proportional delay. The results obtained for non-linear DEs with proportional delay by SAM are compared with exact solutions and other iterative methods. It is noted that SAM is easier to use than other techniques, and the solutions obtained using SAM are consistent with the exact solution.

Keywords. Successive approximation method, Lipschitz condition, Caputo derivative, Existence-uniqueness, Proportional delay, Pantograph equation, Ambartsumian equation.

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1. Introduction

Delay differential equations (DDEs) contain the state variable at a past time $t-\tau$. The inclusion of the delay τ makes the DDE an infinite-dimensional dynamical system. Even if it is very difficult to analyze and solve such equations, this branch is popular among applied scientists due to its applications in various fields.

On the other hand, if the order of the derivative in a differential equation is any arbitrary number (instead of a positive integer), then the equation is called a fractional differential equation (FDE). Even though there are several inequivalent definitions of the fractional derivative operator, one can select the derivative that is appropriate for the model under consideration. This flexibility is a key feature behind the popularity of fractional calculus.

Daftardar-Gejji and coworkers proposed numerical schemes [5, 15] for solving fractional order delay differential equations (FDDE). The modified Laguerre wavelets method [18], spectral collocation method [1], and fractional-order fibonacci-hybrid functions [30] are a few other methods for solving FDDEs. Stability analysis of FDDEs has been proposed in [6–8, 20]. Applications of FDDE are presented in [9, 21, 28, 29].

In general, the delay τ in the DDE $x'(t) = f(t, x(t), x(t-\tau))$ is not constant. The analysis becomes more difficult when τ depends on time or state. The proportional delay differential equation $x'(t) = f(t, x(t), x(qt))$ or a pantograph equation is a particular case of time-dependent DDE with $\tau(t) = (1-q)t$. These equations are proposed by Ockendon and Tayler in the seminal work [22] to model the motion of an overhead trolley wire. A few other applications of these equations are discussed in [10, 12]. The Daftardar-Gejji and Jafari method (DJM) is applied in [11] to find analytical

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∗ Corresponding author. Email: prajaktarajmane@gmail.com.

solutions of the pantograph equation. Furthermore, the authors presented various relationships between the solution series and existing special functions. Patade and Bhalekar proposed the power series solution for the Ambartsumian equation [23] using DJM. The analytical solution for the pantograph equation is discussed in [24]. Vidhyaa *et al.* [34] have obtained oscillation conditions for non-canonical second-order nonlinear delay difference equations with a super-linear neutral term. The asymptotic behavior of third-order delay difference equations with a negative middle term is discussed in [31].

Solving nonlinear FDEs with proportional delay is an important task in mathematical analysis and applications. This motivates us to work on finding solutions to FDEs with proportional delay. In this paper, we derive the existence-uniqueness results for FDEs with proportional delay and find the solutions of FDEs with proportional delay using SAM in terms of power series.

The paper is organized as follows: The basic definitions and preliminary results are presented in section 2, and the SAM is discussed in section 2.1. Existence and uniqueness results are described in section 2.3, and stability analysis is presented in section 3. Series solutions of the pantograph equation and the Ambartsumian equation are described in section 4. Results for fractional differential equations (FDEs) and systems of FDEs are developed in sections 5 and 6, respectively. Sections 7 and 8 deal with illustrative examples. Finally, conclusions are summarized in Section 9.

2. Preliminaries and Notations

Definition 2.1. [19] The Riemann-Liouville fractional integral of order $\alpha > 0$ of $f \in C[0,\infty)$ is defined as:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta) d\zeta, \quad t > 0. \tag{2.1}$$

Definition 2.2. [19] The (left sided) Caputo fractional derivative of $f, f \in C^m_{-1}, m \in \mathbb{N} \cup \{0\}$, is defined as:

$$\begin{aligned} D^{\alpha} f(t) &= \frac{d^m}{dt^m} f(t), \quad \alpha = m, \\ &= I^{m-\alpha} \frac{d^m}{dt^m} f(t), \quad m-1 < \alpha < m, \quad m \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Note that for $0 \leq m-1 < \alpha \leq m$ and $\beta > -1$

$$\begin{aligned} I^{\alpha} x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \\ \left(I^{\alpha} D^{\alpha} f\right)(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}. \end{aligned} \tag{2.3}$$

Definition 2.3. [19] The Mittag-Leffler function is defined as

$$E_{\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \tag{2.4}$$

Definition 2.4. [19] The multi-parameter Mittag-Leffler function is defined as:

$$E_{(\alpha_1,\cdots,\alpha_n),\beta}(z_1, z_2, \cdots, z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_j \geq 0}} (k; l_1, \cdots, l_n) \left[\frac{\prod_{j=1}^{n} z_j^{l_j}}{\Gamma(\beta + \sum_{j=1}^{n} \alpha_j l_j)} \right].$$

where, $(k; l_1, l_2, \cdots, l_n)$ is the multinomial coefficient defined as

$$(k; l_1, l_2, \cdots, l_n) = \frac{k!}{l_1! l_2! \cdots l_n!}. \tag{2.5}$$

Definition 2.5. [16] Consider the DDE,

$$y'(t) = f(y(t), y(t-\tau(t))), \tag{2.6}$$

where $f : R \times R \to R$. The flow $\phi_t(t_0)$ is a solution $y(t)$ of Eq.(2.6) with initial condition $y(t) = t_0$, $t \leq 0$. The point y^* is called equilibrium solution of Eq. (2.6) if $f(y^*, y^*) = 0$.

(a) If, for any $\epsilon > 0$, there exist $\delta > 0$ such that $|t_0 - y^*| < \delta \Rightarrow |\phi_t(t_0) - y^*| < \epsilon$, then the system Eq. (2.6) is stable (in the Lyapunov sense) at the equilibrium y^*.

(b) If the system (2.6) is stable at y^* and moreover, $\lim_{t\to\infty} |\phi_t(t_0) - y^*| = 0$ then the system (2.6) is said to be asymptotically stable at y^*.

2.1. **Successive Approximation Method (SAM).** The successive approximations method (SAM) is a familiar classical technique for solving integral equations [13]. SAM has applications in various fields, including physics, engineering, and applied mathematics, especially in problems involving integral equations arising in initial value problems [14, 32, 33]. Consider the differential equation

$$y'(t) = f(t, y(t)), y(0) = y_0, \tag{2.7}$$

Let $\phi_0(t) = y_0$ be the first approximate solution of the IVP (2.7). Then

$$\phi_1(t) = y_0 + \int_0^t f(x, \phi_0(x))dx,$$
$$\phi_2(t) = y_0 + \int_0^t f(x, \phi_1(x))dx.$$

Continuing in this way, we obtain

$$\phi_{k+1}(t) = y_0 + \int_0^t f(x, \phi_k(x))dx, \quad k = 0, 1, 2, \cdots . \tag{2.8}$$

2.2. **SAM for differential equations with proportional delay:** Consider the differential equation with proportional delay

$$y'(t) = f(t, y(t), y(qt)), y(0) = y_0, 0 < q < 1, \tag{2.9}$$

where f is a continuous function defined on some 3-dimensional rectangle

$$R = \{|t| \leq a, |y(t) - y_0| \leq b, |y(qt) - y_0| \leq b, a > 0, \, b > 0\}.$$

Let $\phi_0(t) = y_0$ be the initial approximation for the IVP (2.9). Then

$$\phi_1(t) = y_0 + \int_0^t f(x, \phi_0(x), \phi_0(qx))dx,$$
$$\phi_2(t) = y_0 + \int_0^t f(x, \phi_1(x), \phi_1(qx))dx.$$

Continuing in this way, we obtain:

$$\phi_{k+1}(t) = y_0 + \int_0^t f(x, \phi_k(x), \phi_k(qx))dx, \qquad k = 0, 1, 2, \cdots . \tag{2.10}$$

2.3. **Existence and uniqueness results.**

Theorem 2.6. *A function ϕ is a solution of the IVP (2.9) on an interval I if and only if it satisfies the integral equation:*

$$y(t) = y_0 + \int_0^t f(x, y(x), y(qx))dx, \quad for \ t \in I. \tag{2.11}$$

Proof. Let ϕ be a solution of the IVP (2.9) on an interval I. Then

$$\phi'(t) = f(t, \phi(t), \phi(qt)), \phi(0) = y_0, \qquad 0 < q < 1 \tag{2.12}$$

The equivalent integral Equation (2.12) is

$$\phi(t) = \phi(0) + \int_0^t f(x, \phi(x), \phi(qx))dx. \tag{2.13}$$

and $\phi(0) = y_0$. Thus ϕ is a solution of the IVP (2.11).

Conversely, suppose Equation (2.13) holds. Differentiating Equation (2.13) w.r.t. t, we get

$$\phi'(t) = f(t, \phi(t), \phi(qt)), 0 < q < 1 \quad \forall t \in I.$$

From Equation (2.11), $\phi(0) = y_0$. Hence, ϕ is a solution of the IVP (2.9). □

Theorem 2.7. *Let f be continuous and $|f| \leq M$ on R. The successive approximation (2.10) exist and are continuous on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\{a, \frac{b}{M}\}$. If $t \in I$ then $(t, y(t), y(qt)) \in R$ and $|\phi_k(t) - y_0| \leq M|t|$, $|\phi_k(qt) - y_0| \leq M|t|$.*

Proof. We prove the result by mathematical induction.

(i) Clearly $\phi(0) = y_0$ is continuous on I. Thus, the theorem is true for $k = 0$.

(ii) For $k = 1$, we have

$$\phi_1(t) = y_0 + \int_0^t f(x, \phi_0(x), \phi_0(qx))dx,$$

$$\phi_1(t) = y_0 + \int_0^t f(x, y_0, y_0)dx.$$

Since f is continuous, $\phi_1(t)$ exists.

$$\begin{aligned}
|\phi_1(t) - y_0| &= |\int_0^t f(x, \phi_0(x), \phi_0(qx))dx| \\
&\leq \int_0^t |f(x, \phi_0(x), \phi_0(qx))|dx \\
&\leq M|t| \\
&\leq b, \quad t \in I, \\
\text{and} \quad |\phi_1(qt) - y_0| &\leq M|qt| \\
&\leq M|t|, \quad 0 < q < 1 \\
&\leq b, \quad t \in I.
\end{aligned}$$

Thus, for $t \in I$, $(t, y(t), y(qt)) \in R$ and $|\phi_1(t) - y_0| \leq M|t|$, $|\phi_1(qt) - y_0| \leq M|t|$.
Thus, the theorem is true for $k = 1$:

(iii) Assume that the theorem is true for $k = n$. i.e. For $t \in I$, $(t, y(t), y(qt)) \in R$ and $|\phi_n(t) - y_0| \leq M|t|$, $|\phi_n(qt) - y_0| \leq M|t|$.

(iv) To prove the theorem for $k = n + 1$.

If $t \in I$, then

$$\phi_{n+1}(t) = y_0 + \int_0^t f(x, \phi_n(x), \phi_n(qx))dx.$$

Since f is continuous, $\phi_{n+1}(t)$ exists on I.

$$\begin{aligned}
|\phi_{n+1}(t) - y_0| &\leq M|t| \\
&\leq b, \quad t \in I, \\
\text{and} \quad |\phi_{n+1}(qt) - y_0| &\leq M|qt|
\end{aligned}$$

$$\leq M|t|, \quad 0 < q < 1$$
$$\leq b, \quad t \in I.$$

Thus, if $t \in I$, $(t, y(t), y(qt)) \in R$ and $|\phi_{n+1}(t) - y_0| \leq M|t|$, $|\phi_{n+1}(qt) - y_0| \leq M|t|$.

Hence, by mathematical induction, the result is true for all positive integer n. □

Theorem 2.8. *(Existence Theorem) Let f be continuous and $|f| \leq M$ on the 3-dimensional rectangle*

$$R = \{|t| \leq a, |y(t) - y_0| \leq b, |y(qt) - y_0| \leq b, a > 0,\ b > 0.\}$$

Suppose f satisfies the Lipschitz condition in its second and third variables with Lipschitz constants L_1 and L_2 such that

$$|f(t, y_1(t), y_1(qt)) - f(t, y_2(t), y_2(qt))| \leq L_1|y_1(t) - y_2(t)| + L_2|y_1(qt) - y_2(qt)|.$$

Then the successive approximations (2.10) converge on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\left\{a, \frac{b}{M}\right\}$ to a solution ϕ of the IVP (2.9) on I.

Proof. We have

$$\phi_k(t) = \phi_0(t) + \sum_{n=1}^{k}[\phi_n(t) - \phi_{n-1}(t)].$$

To prove that the sequence $\{\phi_k\}$ converges, it is enough to prove that the series

$$\phi_0(t) + \sum_{n=1}^{\infty}[\phi_n(t) - \phi_{n-1}(t)] \tag{2.14}$$

is convergent.

By Theorem 2.7 the function ϕ_k all exist and are continuous on I. Also, $|\phi_1(t) - \phi_0(t)| \leq M|t|$ and $|\phi_1(qt) - \phi_0(qt)| \leq M|t|$ for $t \in I$.

Now,

$$\phi_2(t) - \phi_1(t) = \int_0^t [f(x, \phi_1(x), \phi_1(qx)) - f(x, \phi_0(x), \phi_0(qx))]dx$$
$$\therefore |\phi_2(t) - \phi_1(t)| \leq \int_0^t |f(x, \phi_1(x), \phi_1(qx)) - f(x, \phi_0(x), \phi_0(qx))|dx$$
$$\leq \int_0^t [L_1|\phi_1(x) - \phi_0(x)| + L_2|\phi_1(qx) - \phi_0(qx)|]dx$$
$$\leq M(L_1 + L_2)\frac{|t|^2}{2}.$$

We shall prove by mathematical induction

$$|\phi_n(t) - \phi_{n-1}(t)| \leq M(L_1 + L_2)^{n-1}\frac{|t|^n}{n!}. \tag{2.15}$$

We have proven that Equation (2.15) holds for $n = 1, 2$. Assume that (2.15) holds for $n = m$.

We have

$$\phi_{m+1}(t) - \phi_m(t) = \int_0^t [f(x, \phi_m(x), \phi_m(qx)) - f(x, \phi_{m-1}(x), \phi_{m-1}(qx))]dx$$
$$\therefore |\phi_{m+1}(t) - \phi_m(t)| \leq \int_0^t |f(x, \phi_m(x), \phi_m(qx)) - f(x, \phi_{m-1}(x), \phi_{m-1}(qx)|dx$$
$$\leq \int_0^t [L_1|\phi_m(x) - \phi_{m-1}(x)| + L_2|\phi_m(qx) - \phi_{m-1}(qx)|]dx$$

$$\leq M(L_1+L_2)^m \frac{|t|^{m+1}}{(m+1)!}..$$

Thus, the result is true for $n = m + 1$.

Hence, by the mathematical induction, the result is true for all $n = 1, 2, \cdots$. Therefore, the infinite series (2.15) is absolutely convergent on I. This shows that the n^{th} term of the series $|\phi_0(t)| + \sum_{n=1}^{\infty} |\phi_n(t) - \phi_{n-1}(t)|$ is less than $\frac{M}{(L_1+L_2)}$ times the n^{th} term of the power series $e^{(L_1+L_2)|t|}$. Hence, the series (2.15) is converges. □

3. Stability Analysis

The differential equations with proportional delay

$$y'(t) = f\left(t, y(t), y(qt)\right), \tag{3.1}$$

is a special case of the time-dependent delay differential equation (DDE)

$$y'(t) = f\left(t, y(t), y\left(t - \tau(t)\right)\right) \quad \text{with} \quad \tau(t) = (1-q)t,$$

The following results are similar to those in [16].

Theorem 3.1. *Suppose that the equilibrium solution y^* of the equation*

$$y' = f(y(t), y(t-\tau^*)), \quad \tau^* = \tau(t_0), \tag{3.2}$$

is stable and $\|f(y(t), y(t-\tau(t))) - f(y(t), y(t-\tau(t_1)))\| < \epsilon_1 |t - t_1|$, for some $\epsilon_1 > 0$ and for all $t,\ t_1 \in [t_0, t_0 + c)$, where c is a positive constant. Then there exists $\bar{t} > 0$ such that the equilibrium solution y^* of (2.6) is stable on the finite time interval $[t_0, \bar{t})$.

Corollary 3.2. *If the real parts of all roots of $\lambda - a - be^{-\lambda\tau^*} = 0$ are negative, where $a = \partial_1 f, b = \partial_2 f$ are evaluated at the equilibrium, then there exist $\epsilon_c, \bar{t}(> t_0)$, such that when $\epsilon_1 < \epsilon_c$, the solution $y^* = 0$ of (2.6) is stable on the finite time interval $[t_0, \bar{t})$.*

4. Series Solution of Pantograph Equation

The pantograph is a device used in electric trains to collect current from overhead lines. The pantograph equation was formulated by Ockendon and Taylor in 1971 and originates in electrodynamics [22].

Consider the pantograph equation:

$$y'(t) = ay(t) + by(qt), \quad y(0) = 1, \tag{4.1}$$

where $0 < q < 1$, $a, b \in R$. Integrating (4.1), we get

$$y(t) = 1 + \int_0^t \left(ay(x) + by(qx)\right) dt \tag{4.2}$$

Let $\phi_k(t)$ denote the k^{th} approximate solution, with the initial approximate

$$\phi_0(t) = 1. \tag{4.3}$$

For $k \geq 1$, the recurrent formula as below:

$$\phi_k(t) = 1 + \int_0^t \left(a\phi_{k-1}(x) + b\phi_{k-1}(qx)\right) dx. \tag{4.4}$$

From this recurrence relation, we have

$$\begin{aligned} \phi_1(t) &= 1 + \int_0^t \left(a\phi_0(x) + by_0(qx)\right) dx \\ &= 1 + (a+b)\frac{t}{1!}, \\ \phi_2(t) &= 1 + \int_0^t \left(a\phi_1(x) + b\phi_1(qx)\right) dx \end{aligned}$$

$$= 1 + (a+b)\frac{t}{1!} + (a+b)(a+bq)\frac{t^2}{2!},$$

$$\phi_3(t) = 1 + \int_0^t (a\phi_2(x) + b\phi_2(qx))\,dx$$

$$= 1 + (a+b)\frac{t}{1!} + (a+b)(a+bq)\frac{t^2}{2!} + (a+b)(a+bq)(a+bq^2)\frac{t^3}{3!},$$

$$\vdots$$

$$\phi_k(x) = 1 + \frac{t^k}{k!}\prod_{j=0}^{k-1}\left(a+bq^j\right), \quad k = 1,2,3\cdots.$$

$$\text{As} \quad k \to \infty, \quad \phi_k(t) \to y(t)$$

$$y(t) = 1 + \sum_{m=1}^{\infty}\frac{t^m}{m!}\prod_{j=0}^{m-1}\left(a+bq^j\right).$$

If we define $\prod_{j=0}^{m-1}\left(a+bq^j\right) = 1$, for $m = 0$, then

$$y(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\prod_{j=0}^{m-1}\left(a+bq^j\right). \tag{4.5}$$

Theorem 4.1. *For $0 < q < 1$, the power series (4.5) is convergent for all $t \in R$.*

Corollary 4.2. *The power series (4.5) is absolutely convergent for all t and hence it is uniformly convergent on any compact interval of R.*

Theorem 4.3. *If $0 < q < 1$ and $a, b \geq 0$, then*

$$e^{at} \leq y(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\prod_{j=0}^{m-1}\left(a+bq^j\right) \leq e^{(a+b+c)t}, \quad 0 \leq t < \infty.$$

Theorem 4.4. *If $(a+b) < 0$ then the zero solution of Eq. (4.1) is asymptotically stable.*

Proof. Define

$$u(t) = \max_{0\leq x\leq t} y^2(t)$$

$$\therefore \frac{1}{2}u'(t) = \frac{1}{2}\frac{d}{dt}(y^2(t))$$

$$= y(t)y'(t)$$

$$= y(t)(ay(t) + by(qt))$$

$$= ay^2(t) + by(t)y(qt)$$

$$\leq (a+b)u(t)$$

$$\Rightarrow u(t) \leq u(0)e^{2(a+b)t}$$

$$\therefore \lim_{x\to\infty} y(t) = 0, \quad \text{if} \quad (a+b) < 0.$$

□

4.1. **Series solution of Ambartsumian equation.** In [3], Ambartsumian derived a delay differential equation describing the fluctuations of the surface brightness in a the Milky Way. The equation is given as:

$$y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right), \tag{4.6}$$

where $q > 1$ is a constant for the given model.

Equation (4.6) with the initial condition $y(0) = \lambda$ can be written equivalently as

$$y(t) = \lambda + \int_0^t \left(\frac{1}{q} y\left(\frac{x}{q}\right) - y(x) \right) dx. \tag{4.7}$$

Let $\phi_k(t)$ be the k^{th} approximate solution, where the initial approximate solution is taken as

$$\phi_0(t) = \lambda. \tag{4.8}$$

For $k \geq 1$, we use the recurrence relation:

$$\phi_k(t) = \lambda + \int_0^t \left(\frac{1}{q} \phi_{k-1}\left(\frac{x}{q}\right) - \phi_{k-1}(x) \right) dx. \tag{4.9}$$

From this recurrence relation, we have:

$$\begin{aligned}
\phi_1(t) &= \lambda + \int_0^t \left(\frac{1}{q} \phi_0\left(\frac{x}{q}\right) - \phi_0(x) \right) dx \\
&= \lambda + \int_0^t \left(\frac{\lambda}{q} - \lambda \right) dx \\
&= \lambda + \left(\frac{\lambda}{q} - \lambda \right) \frac{t}{1!} \\
&= \left(1 + \left(\frac{1}{q} - 1 \right) \frac{t}{1!} \right) \lambda, \\
\phi_2(t) &= \lambda + \int_0^t \left(\frac{1}{q} \phi_1\left(\frac{x}{q}\right) - \phi_1(x) \right) dx \\
&= \left(1 + \left(\frac{1}{q} - 1 \right) \frac{t}{1!} + \left(\frac{1}{q} - 1 \right)\left(\frac{1}{q^2} - 1 \right) \frac{t^2}{2!} \right) \lambda, \\
&\vdots \\
\phi_k(t) &= \left(1 + \sum_{m=1}^{k} \frac{t^m}{m!} \prod_{j=1}^{m} \left(\frac{1}{q^j} - 1 \right) \right) \lambda. \\
\text{As} \quad & k \to \infty, \quad \phi_k(t) \to y(t) \\
y(t) &= \left(1 + \sum_{m=1}^{\infty} \frac{t^m}{m!} \prod_{j=1}^{m} \left(\frac{1}{q^j} - 1 \right) \right) \lambda.
\end{aligned}$$

If we define $\prod_{j=1}^{m} \left(\frac{1}{q^j} - 1 \right) = 1$, for $m = 0$, then

$$y(t) = \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \prod_{j=1}^{m} \left(\frac{1}{q^j} - 1 \right) \right) \lambda. \tag{4.10}$$

Theorem 4.5. *For $q > 1$, the power series (4.10) is convergent for all $t \in R$.*

Corollary 4.6. *The power series (4.10) is absolutely convergent for all t and is therefore uniformly convergent on any compact interval of R.*

Theorem 4.7. *The zero solution of (4.6) is asymptotically stable.*

5. Fractional order differential equations with proportional delay

Consider the initial value problem (IVP)

$$\begin{aligned} D^{\alpha}y(t) &= f(t, y(t), y(qt)), \quad 0<\alpha\le 1,\ 0<q<1, \\ y(0) &= y_0, \end{aligned} \tag{5.1}$$

where D^{α} denotes the Caputo fractional derivative and f is a continuous function defined on the 3-dimensional rectangle

$$R = \{|t|\le a,\ |y(t)-y_0|\le b,\ |y(qt)-y_0|\le b,\ a>0,\ b>0\}.$$

Theorem 5.1. *A function ϕ is a solution of the IVP (5.1) on an interval I if and only if it is a solution of the integral equation*

$$y(t) = y_0 + \int_0^t \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x, y(x), y(qx))dx, \quad on \quad I. \tag{5.2}$$

Theorem 5.2. *Let f be continuous and $|f|\le M$ on R. The successive approximation*

$$\begin{aligned} \phi_{k+1}(t) &= y_0, \\ \phi_{k+1}(t) &= y_0 + \int_0^t \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x, \phi_k(x), \phi_k(qx))dx, \quad k=0,1,2,\cdots. \end{aligned} \tag{5.3}$$

exist and are continuous on the interval $I=[-\zeta,\zeta]$, where $\zeta = min\left\{a, (\frac{\Gamma(\alpha+1)b}{M})^{\frac{1}{\alpha}}\right\}$. If $t\in I$ then $(t, y(t), y(qt))\in R$ and $|\phi_k(t)-y_0|\le M\frac{|t|^{\alpha}}{\Gamma(\alpha+1)}$, $|\phi_k(qt)-y_0|\le M\frac{|t|^{\alpha}}{\Gamma(\alpha+1)}$.

Theorem 5.3. *(Existence Theorem) Let f be continuous and $|f|\le M$ on the 3-dimensional rectangle*

$$R = \{|t|\le a,\ |y(t)-y_0|\le b,\ |y(qt)-y_0|\le b,\ a>0,\ b>0\}.$$

Suppose f satisfies the Lipschitz condition in its second and third variable with Lipschitz constants L_1 and L_2 such that

$$|f(t, y_1(t), y_1(qt)) - f(t, y_2(t), y_2(qt))| \le L_1|y_1(t)-y_2(t)| + L_1|y_1(qt)-y_2(qt)|.$$

Then the successive approximations (5.3) converge on the interval $I=[-\zeta,\zeta]$, where $\zeta = min\left\{a, (\frac{\Gamma(\alpha+1)b}{M})^{\frac{1}{\alpha}}\right\}$ to a solution ϕ of the IVP (5.1) on I.

5.1. Series solution of fractional order Pantograph equation.

Consider the fractional order pantograph equation as:

$$D^{\alpha}y(t) = ay(t) + by(qt), \quad y(0)=1, \tag{5.4}$$

where $0<\alpha\le 1$, $0<q<1$, $a, b\in R$.

The solution of (5.4) using successive approximation is

$$y(t) = \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m+1)} \prod_{j=0}^{m-1}\left(a+bq^{\alpha j}\right). \tag{5.5}$$

Theorem 5.4. *If $0<q<1$, then the power series (5.5) is convergent for all finite values of t.*

Theorem 5.5. *If $0<q<1$ and $a, b\ge 0$, then*

$$E_{\alpha}(at^{\alpha}) \le y(t) = \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m+1)} \prod_{j=0}^{m-1}\left(a+bq^{\alpha j}\right) \le E_{\alpha}((a+b)t^{\alpha}), \quad 0\le t<\infty.$$

5.2. **Series solution of fractional order Ambartsumian equation.** Consider the fractional order Ambartsumian equation as:

$$D^{\alpha}y(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right), \quad y(0) = 1, \tag{5.6}$$

where $q > 1$ and is constant for the given model.

The solution of (5.6) using successive approximation is

$$y(t) = \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m + 1)} \prod_{j=0}^{m-1} \left(\frac{1}{q^{1+\alpha j}} - 1\right). \tag{5.7}$$

Theorem 5.6. *If $q > 1$, then the power series (5.7) is convergent for all finite values of t.*

6. System of fractional order differential equations with proportional delay

Consider the initial value problem (IVP):

$$\begin{aligned} &D^{\alpha_i}y_i(t) = f_i(t, \bar{y}(t), \bar{y}(qt)), \quad 0 < \alpha_i \leq 1, \ 0 < q < 1, \\ &y_i(0) = {}^i y_0, \quad 1 \leq i \leq n, \end{aligned} \tag{6.1}$$

where D^{α_i} denotes the Caputo fractional derivative, $\bar{y}(t) = (y_1(t), y_2(t) \cdots, y_n(t))$, $\bar{y}(qt) = (y_1(qt), y_2(qt) \cdots, y_n(qt))$ and $f = (f_1, f_2 \cdots, f_n)$ is a continuous function defined on the $(2n + 1)$ dimensional rectangle

$$R = \{|t| \leq a, |y_i(t) -^i y_0| \leq b_i, |y_i(qt) -^i y_0| \leq b_i, a > 0, \ b_i > 0, 1 \leq i \leq n\}.$$

Theorem 6.1. *A function $\bar{\phi}$ is a solution of the IVP (6.1) on an interval I if and only if it is a solution of the integral equation*

$$y_i(t) =^i y_0 + \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f(x, \bar{y}(x), \bar{y}(qx))dx \quad on \quad I, \tag{6.2}$$

where $\bar{\phi}_m = \left({}^1\phi_m, {}^2\phi_m, \cdots, {}^n\phi_m\right)$

Theorem 6.2. *Let $||f|| = M$ on rectangle R. The successive approximation*

$$\begin{aligned} &{}^i\phi_0(t) =^i y_0, \quad i = 0, 1, 2, \cdots. \\ &{}^i\phi_{k+1}(t) = y_0 + \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f(x, \bar{\phi}_k(x), \bar{\phi}_k(qx))dx. \quad k = 0, 1, 2, \cdots. \end{aligned} \tag{6.3}$$

exist and are continuous on the interval $I = [-\zeta, \zeta]$, where

$$\zeta = min\left\{a, \left(\frac{\Gamma(\alpha_1+1)b_1}{M}\right)^{\frac{1}{\alpha_1}}, \cdots, \left(\frac{\Gamma(\alpha_n+1)b_n}{M}\right)^{\frac{1}{\alpha_n}}\right\}.$$

If t is in the interval I then $(t, \bar{y}_m(t), \bar{y}_m(qt))$ is in rectangle R and $||\bar{y}_m(t) - \bar{y}(0)|| \leq M \sum_{i=1}^{m} \frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}$, $||\bar{y}_m(qt) - \bar{y}(0)|| \leq M \sum_{i=1}^{m} \frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}$ $\forall m$.

Theorem 6.3. *Let f be a continuous function defined on the rectangle*

$$R = \{|t| \leq a, |y_i(t) -^i y_0| \leq b_i, |y_i(qt) -^i y_0| \leq b_i, a > 0, \ b_i > 0, 1 \leq i \leq n\}.$$

Suppose f satisfies Lipschitz condition in second and third variable with Lipschitz constants L_1 and L_2 such that $|f(t, \bar{y_1}(t), \bar{y_1}(qt)) - f(t, \bar{y_2}(t), \bar{y_2}(qt))| \leq L_1|\bar{y}_1(t) - \bar{y}_2(t)| + L_1|\bar{y}_1(qt) - \bar{y}_2(qt)|$. Then the successive approximations (6.3) converges on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\left\{a, \left(\frac{\Gamma(\alpha_1+1)b_1}{M}\right)^{\frac{1}{\alpha_1}}, \cdots, \left(\frac{\Gamma(\alpha_n+1)b_n}{M}\right)^{\frac{1}{\alpha_n}}\right\}$ to a solution of the ϕ of the IVP (6.1) on I.

6.1. **System of fractional-order Pantograph equation.** Consider the system of fractional-order pantograph equation

$$D^\alpha y(t) = Ay(t) + By(qt), \quad y(0) = y_0, \quad 0 < \alpha \le 1, \tag{6.4}$$

where $0 < q < 1$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, and $y = [y_1, y_2, \cdots, y_n]^T$.

The solution of (4.4) using successive approximation is

$$y(t) = \left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}(A + Bq^{-(k-j)\alpha})\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right]\lambda. \tag{6.5}$$

Theorem 6.4. *For $0 < q < 1$, the power series (6.5) is convergent for $t \in R$.*

6.2. **System of fractional-order Ambartsumian equations.** In this section, we generalize the Ambartsumian Equation (2.9) to the system of fractional-order Ambartsumian equations [25] as:

$$D^\alpha y(t) = -Iy(t) + By\left(\frac{t}{q}\right), \quad y(0) = \lambda, \quad 0 < \alpha \le 1, \tag{6.6}$$

where D^α denotes the Caputo fractional derivative, I is the identity matrix of order n and $1 < q$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$,

and $B = \begin{bmatrix} \frac{1}{q} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{q} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{q} \end{bmatrix}_{n\times n}$.

Applying SAM to the initial value problem (6.6), we have

$$y(t) = y(0) - IJ^\alpha y(t) + BJ^\alpha y\left(\frac{t}{q}\right). \tag{6.7}$$

Let $\phi_k(t)$ denote the kth approximate solution, where the initial approximate solution is taken as

$$\phi_0(t) = \lambda. \tag{6.8}$$

For $k \ge 1$, we use the following recurrence relation:

$$\phi_k(t) = \lambda - IJ^\alpha \phi_{k-1}(t) + BJ^\alpha \phi_{k-1}\left(\frac{t}{q}\right). \tag{6.9}$$

From this recurrence relation, we derive:

$$\begin{aligned}
\phi_1(t) &= \lambda - IJ^\alpha\phi_0(t) + BJ^\alpha\phi_0\left(\frac{t}{q}\right) \\
&= \lambda - I\frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + B\frac{\lambda t^\alpha}{\Gamma(\alpha+1)} \\
&= \left(I + (-I+B)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\lambda, \\
\phi_2(t) &= \lambda - IJ^\alpha\phi_1(t) + BJ^\alpha\phi_1\left(\frac{t}{q}\right) \\
&= \lambda - IJ^\alpha\left[\left(I + (-I+B)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\lambda\right] + BJ^\alpha\left(I + (-I+B)\frac{q^{-\alpha}t^\alpha}{\Gamma(\alpha+1)}\right)\lambda
\end{aligned}$$

$$= \lambda - I\left[\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + (-I+B)\frac{\lambda t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + B\left[\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + (-I+B)\frac{\lambda q^{-\alpha} t^{2\alpha}}{\Gamma(2\alpha+1)}\right]$$
$$= \left[I + (-I+B)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (-I+Bq^{-\alpha})(-I+B)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\lambda,$$

$$\phi_3(t) = \Bigg[I + (-I+B)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (-I+Bq^{-\alpha})(-I+B)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+(-I+Bq^{-2\alpha})(-I+Bq^{-\alpha})(-I+B)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Bigg]\lambda,$$
$$\ldots$$
$$\phi_k(t) = \left[I + \sum_{m=1}^{k}\prod_{j=1}^{m}(-I+Bq^{-(m-j)\alpha})\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\right]\lambda.$$

As $k \to \infty$, $\quad \phi_k(t) \to y(t)$

$$y(t) = \left[I + \sum_{k=1}^{\infty}\prod_{j=1}^{k}(-I+Bq^{-(k-j)\alpha})\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right]\lambda.$$

If we set $\prod_{j=1}^{k}(-I+Bq^{(k-j)\alpha}) = I$, for $k=0$, then

$$y(t) = \left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}(-I+Bq^{-(k-j)\alpha})\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right]\lambda. \tag{6.10}$$

Theorem 6.5. *For $q > 1$, the power series*

$$y(t) = \left[\sum_{k=0}^{\infty}\prod_{j=1}^{k}(-I+Bq^{-(k-j)\alpha})\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right]\lambda,$$

is convergent for $t \in R$.

Proof. The result follows immediately from the ratio test [4]. □

7. Illustrative Examples

Example 7.1. Consider the nonlinear differential equations with proportional delay [2, 17, 26, 27]

$$\frac{dy(t)}{dt} = 1 - 2y^2\left(\frac{t}{2}\right), \quad y(0) = 0. \tag{7.1}$$

The corresponding integral equation is:

$$y(t) = \int_0^t \left(1 - 2u^2\left(\frac{t}{2}\right)\right)dx. \tag{7.2}$$

Using the successive approximation method (2.10), we obtain:

$$\phi_0(t) = 0,$$
$$\phi_1(t) = t,$$
$$\phi_2(t) = t - \frac{t^3}{6},$$
$$\phi_3(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{8064},$$

$$\phi_4(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{61t^9}{23224320} - \frac{67t^{11}}{3406233600} + \frac{t^{13}}{12881756160} - \frac{t^{15}}{7990652436480},$$
$$\phi_5(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{61t^9}{23224320} - \cdots - \frac{t^{31}}{106266419988615169375835859588218800},$$

and so on.

The exact solution of (7.1) is $y(t) = \sin t$. The 5-term solutions obtained via:

- Adomian decomposition method (ADM) [17]
- Variational iteration method (VIM) [26]
- Homotopy analysis method (HAM) [2]
- Optimal homotopy asymptotic method (OHAM) [27]

all yield:

$$\begin{aligned} y(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}}{39916800} + \frac{t^{13}}{6227020800} \\ &\quad - \frac{t^{15}}{1307674368000} + \frac{t^{17}}{355687428096000}. \end{aligned} \tag{7.3}$$

The 4-term OHAM solution [27] of (7.1) is:

$$y(t) = t - 0.166665t^3 + 0.00832857t^5 - 0.000192105t^7. \tag{7.4}$$

We compare 5^{th} approximation solution (SAM) and 5-term solutions (ADM, VIM, HAM) with exact solution in Figure 1 and 4^{th} approximation solution (SAM) with 4-term solution (OHAM) in Figure 2. The absolute errors in computation are shown in Figures 3–4. It can be observed that SAM solution is better than the solution obtained using other methods.

8. Figures

Remark 8.1. While the ADM, VIM, HAM, and OHAM solutions presented in [2, 17, 26, 27] were limited to the interval $[0, 1]$, our SAM approach has successfully extended the solution to the wider interval $[0, 8]$.

9. Conclusions

Using SAM, we have successfully solved nonlinear differential equations with proportional delay. Our analysis has established stability, uniqueness, and existence results for several specific types of time-dependent DDEs. Furthermore, we have examined the convergence of series solutions for both pantograph and Ambartsumian equations. The integer-order analysis developed for DEs with proportional delay and time-dependent DDEs has been successfully extended to the fractional-order case using the Caputo derivative. Numerical examples validate all theoretical results.

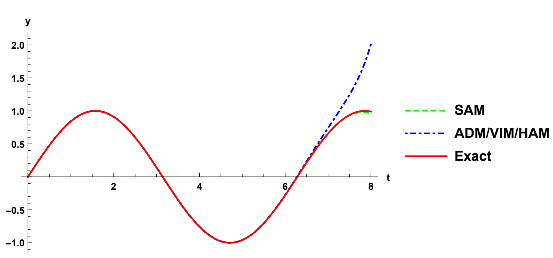

Figure 1. Comparison of SAM, ADM, VIM, and HAM solutions with the exact solution of Equation (7.1).

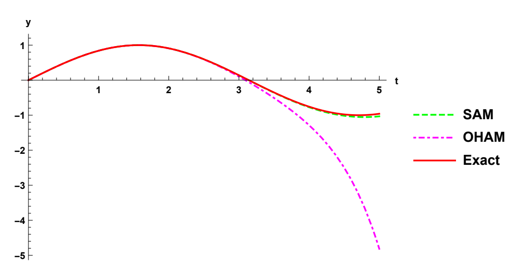

Figure 2. Comparison of SAM and OHAM solutions with exact solution of Equation (7.1).

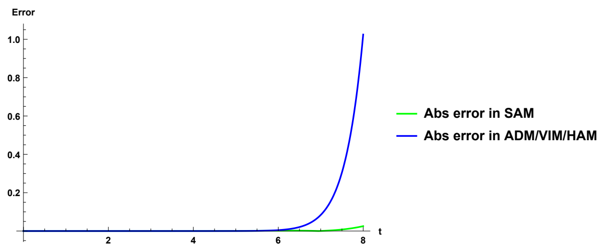

FIGURE 3. Comparison of absolute errors in SAM and ADM/VIM/HAM solutions.

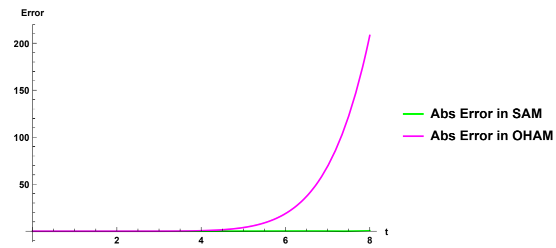

FIGURE 4. Comparison of absolute errors in SAM and OHAM solutions.

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