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Numerical solution of fractional Volterra integro-differential equations using flatlet oblique multiwavelets

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Abstract

The presented paper investigates a new numerical method based on the characteristics of flatlet oblique multiwavelets for solving fractional Volterra integro-differential equations, in this method, first using the dual bases of the flatlet multiwavelets, the operator matrices are made for the derivative of fractional order and Volterra integral. Then, the fractional Volterra integro-differential equation reduces to a set of algebraic equations which can be easily solved. The error analysis and convergence of the presented method are discussed. Also, numerical examples will indicate the acceptable accuracy of the proposed method, which is compared with the methods used by other researchers.

Keywords. Flatlet oblique multiwavelets, Fractional Volterra integro-differential equations, Operational matrix, Collocation method, Biorthogonal system.

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1. Introduction

The wide application of fractional calculus in various sciences and engineering fields such as physics, biophysics, cosmology, bioengineering, control theory, finance and statistical mechanics have attracted the attention of many researchers to investigate different numerical methods for solving fractional calculus problems. Some of the methods that have been used in the works [2–20] to solve the fractional calculus equations include the adomian decomposition method, Bessel collocation method, CAS wavelets method, Chebyshev pseudo-spectral method, cubic B-spline wavelets method, Euler wavelets, and fractional differential transform method. Also Jacobi spectral-collocation method [23, 36], Legendre collocation method [31], multi-domian pseudo-spectral method [22], normalized systems functions method [33], novel Legendre wavelet Petrov-Galerkin method method [32], operational Tau method [19], piecewise polynomial collocation method [37], Taylor expansion [31], and variational iteration methods [13, 24] are other techniques that have been proposed to solve this category of problems.

Flatlet oblique multiwavelets are used for solving integer order integro-differential equations [9]. Also, in 2014, Dr. Darani et al [8] solved the fractional integro-differential equation by constructing dual scale and wavelet functions in the form of fractional degree polynomials for the Flatlet multiwavelets. However, since no method based on Flatlet oblique multiwavelets has been proposed to solve the fractional Volterra integro-differential equation, in this paper, we have solved this problem using the fractional order derivative and Volterra integration operator matrices constructed by the basis of the flatlet multiwavelets.

The general form of the linear fractional integro-differential equation is expressed as

$$D^{\alpha}y(x) + q(x)y(x) = g(x) + \lambda \int_0^x K(x,t)y(t)dt, \tag{1.1}$$

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with initial conditions

$$y^j(0) = c_j, \qquad j = 0, 1, ..., n-1, \quad n-1 < \alpha < n,$$

where $n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$. $K(x,t)$, $q(x)$, and $g(x)$ are given continuous functions and $y(x)$ is the unknown function to be determined. Also, $D^\alpha y(x)$ indicates the Caputo's fractional derivative of $y(x)$.

Definition 1.1. ([26]) The Caputo fractional differentiation operator D^α of order α is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^n(t)}{(x-t)^{\alpha+1-n}} dt, \quad \alpha > 0,$$

where $n-1 < \alpha < n$ and $n \in \mathbb{Z}^+$.

In this research, in section 2, we will introduce and examine the properties of Flatlet oblique multiwavelets and explain how an arbitrary function can be written as an extension of the scale and mother wavelet functions of Flatlet multiwavelets. Also, we will discuss how to construct dual scale and mother wavelet functions using the biorthogonal property of Flatlet multiwavelets in section 2. Section 3 includes the construction of Volterra integration and fractional derivative operator matrices, as well as writing the fractional Volterra integro-differential equation as a set of algebraic equations and solving this set of equations. The convergence and error analysis of the proposed method will be discussed in section 4. Finally, in section 5, some numerical examples are presented to show the applicability and accuracy of the proposed method.

2. Flatlet Multiwavelet System and the Duale Functions

functions $\phi_0(x), ..., \phi_m(x)$ defined by

$$\phi_i(x) = \begin{cases} 1, & \frac{i}{m+1} \leq x < \frac{i+1}{m+1}, \\ 0, & \text{otherwise}, \end{cases} \quad , i = 0, 1, ..., m. \tag{2.1}$$

The simplest member of this family is the Haar wavelet which happens for the case $m = 0$. Each of these unit constant functions is called a scale function. The Flatlet mother wavelets $\psi_0(x), ..., \psi_m(x)$, corresponding to Flatlet scaling functions are constructed by using two-scale relation which will be discussed in the following.

First, consider two vector functions

$$\Phi(x) = \begin{bmatrix} \phi_0(x) \\ \vdots \\ \phi_i(x) \\ \vdots \\ \phi_m(x) \end{bmatrix}, \Psi(x) = \begin{bmatrix} \psi_0(x) \\ \vdots \\ \psi_i(x) \\ \vdots \\ \psi_m(x) \end{bmatrix}, \tag{2.2}$$

whose components are Flatlet scaling functions and mother wavelets, respectively.

The two-scale relations for the Flatlet multiwavelet system are expressed as

$$\Phi(x) = \mathbf{R} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \Psi(x) = \mathbf{S} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{2.3}$$

where $\mathbf{R}$ and $\mathbf{S}$ are $(m+1) \times 2(m+1)$ matrices. The matrix form of two-scale relations (2.3) are as follows

$$\begin{bmatrix} \Phi(x) \\ \Psi(x) \end{bmatrix} = \begin{bmatrix} \mathbf{R} \\ \mathbf{S} \end{bmatrix} \begin{bmatrix} \Phi(2x) \\ \Phi(2x-1) \end{bmatrix}, \tag{2.4}$$

and the coefficient matrix in (2.4) is called reconstruction matrix that is invertible [9]. Also, Eq. (2.1) for $i = 0, \cdots, (m+1)2^J$ can be extended as follows

$$\Phi_i(x) = \sum_{j=1}^{m+1} R_{i,j} \phi_{j-1}(2x) + \sum_{j=m+2}^{2m+2} R_{i,j} \phi_{j-m-2}(2x-1),$$

$$\Psi_i(x) = \sum_{j=1}^{m+1} S_{i,j}\phi_{j-1}(2x) + \sum_{j=m+2}^{2m+2} S_{i,j}\phi_{j-m-2}(2x-1), \tag{2.5}$$

which is called reconstruction relations.

As is clear, the Flatlet scaling functions have a simple form so the matrix $\mathbf{R}$ can be calculated as

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & & & & 0 \\ & & 1 & 1 & & \\ & & & \ddots & & \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

In order to compute $2(m+1)^2$ entries of matrix $\mathbf{S}$, $2(m+1)^2$ independent conditions are needed. For this purpose, $\frac{(m+1)(m+2)}{2}$ orthonormality conditions

$$\int_0^1 \psi_i(x)\psi_j(x)dt = \delta_{i,j}, \qquad i,j = 0,1,...,m, \tag{2.6}$$

and $\frac{(m+1)(3m+2)}{2}$ vanishing moment conditions

$$\int_{-\infty}^{\infty} \psi_i(x)x^j dx = 0, \quad i = 0,1,...,m, \quad j = 0,1,...,m+i, \tag{2.7}$$

come to our aid where $\delta_{i,j}$ is Kronecker delta defined as

$$\delta_{i,j} = \begin{cases} 1 & if \;\; i = j, \\ 0 & if \;\; i \neq j. \end{cases}$$

By using (2.2) and (2.4), Eq. (2.9) can be written as a system of linear equations

$$\sum_{l=0}^{2(m+1)} \{(l+1)^{j+1} - (l)^{j+1}\}s_{j,l} = 0, \qquad j = 0,...,m+i, \quad i = 0,1,\cdots,m. \tag{2.8}$$

Now, the unknown matrix $\mathbf{S}$ and so $\mathbf{\Psi}(x)$ are obtained by solving (2.6)-(2.9). As an example, for the first order Flatlet basis functions we have

$$\phi_0(x) = \left\{ \begin{array}{ll} 1, & 0 \leq x < \frac{1}{2}, \\ 0, & \text{otherwise}, \end{array} \right. \quad , \quad \phi_1(x) = \left\{ \begin{array}{ll} 1, & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise}, \end{array} \right. \tag{2.9}$$

and the matrix $\mathbf{S}$ is computed as

$$\mathbf{S} = \pm \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}. \tag{2.10}$$

This computation implies that the associated multiwavelets are not unique. A simple form of mother wavelets for the above example may be given as

$$\psi_0(x) = \sqrt{2} \left\{ \begin{array}{ll} \frac{1}{2}, & 0 \leq x < \frac{1}{4}, \\ -\frac{1}{2}, & \frac{1}{4} \leq x < \frac{3}{4}, \\ \frac{1}{2}, & \frac{3}{4} \leq x < 1, \\ 0, & \text{otherwise}, \end{array} \right. \quad , \quad \psi_1(x) = \sqrt{10} \left\{ \begin{array}{ll} \frac{1}{10}, & 0 \leq x < \frac{1}{4}, \\ -\frac{3}{10}, & \frac{1}{4} \leq x < \frac{1}{2}, \\ \frac{3}{10}, & \frac{1}{2} \leq x < \frac{3}{4}, \\ -\frac{1}{10}, & \frac{3}{4} \leq x < 1, \\ 0, & \text{otherwise}. \end{array} \right. \tag{2.11}$$

Also the second order Flatlet multiwavelet system can be represented as

$$\phi_i(x) = \left\{ \begin{array}{ll} 1, & \frac{i}{3} \leq x < \frac{i+1}{3}, \\ 0, & \text{otherwise}, \end{array} \right. , i = 0,1,2,$$

$$\psi_0(x)=\sqrt{10}\begin{cases}\frac{1}{6}, & 0\le x<\frac{1}{6},\\ -\frac{7}{30}, & \frac{1}{6}\le x<\frac{1}{3},\\ -\frac{2}{15}, & \frac{1}{3}\le x<\frac{1}{2},\\ \frac{2}{15}, & \frac{1}{2}\le x<\frac{2}{3},\\ \frac{7}{30}, & \frac{2}{3}\le x<\frac{5}{6},\\ -\frac{1}{6}, & \frac{5}{6}\le x<1,\\ 0, & \text{otherwise},\end{cases} \quad , \quad \psi_1(x)=\sqrt{14}\begin{cases}\frac{1}{14}, & 0\le x<\frac{1}{6},\\ -\frac{3}{14}, & \frac{1}{6}\le x<\frac{1}{3},\\ \frac{1}{7}, & \frac{1}{3}\le x<\frac{2}{3},\\ -\frac{3}{14}, & \frac{2}{3}\le x<\frac{5}{6},\\ \frac{1}{14}, & \frac{5}{6}\le x<1,\\ 0, & \text{otherwise},\end{cases}$$

$$\psi_2(x)=\sqrt{14}\begin{cases}-\frac{1}{42}, & 0\le x<\frac{1}{6},\\ \frac{5}{42}, & \frac{1}{6}\le x<\frac{1}{3},\\ -\frac{5}{21}, & \frac{1}{3}\le x<\frac{1}{2},\\ \frac{5}{21}, & \frac{1}{2}\le x<\frac{2}{3},\\ -\frac{5}{42}, & \frac{2}{3}\le x<\frac{5}{6},\\ \frac{1}{42}, & \frac{5}{6}\le x<1,\\ 0, & \text{otherwise},\end{cases} \tag{2.12}$$

2.1. **Biorthogonal Flatlet Multiwavelet System (BFMS).** Here, we introduce the dual scaling and wavelet vector functions in biorthogonal Flatlet multiwavelet system (BFMS) respectively by $\tilde{\Phi}(x)$ and $\tilde{\Psi}(x)$ that are represented as

$$\tilde{\Phi}(x)=\begin{bmatrix}\tilde{\phi_0}(x)\\ \vdots\\ \tilde{\phi_i}(x)\\ \vdots\\ \tilde{\phi_m}(x)\end{bmatrix}, \tilde{\Psi}(x)=\begin{bmatrix}\tilde{\psi_0}(x)\\ \vdots\\ \tilde{\psi_i}(x)\\ \vdots\\ \tilde{\psi_m}(x)\end{bmatrix}. \tag{2.13}$$

According to the biorthogonality conditions we must have

$$\begin{aligned}&\langle\tilde{\phi_i},\phi_j\rangle=\int_0^1\tilde{\phi_i}(x)\phi_j(x)dx=\delta_{i,j},\\ &\langle\tilde{\psi_i},\psi_j\rangle=\int_0^1\tilde{\psi_i}(x)\psi_j(x)dx=\delta_{i,j},\\ &\langle\tilde{\psi_i},\phi_j\rangle=\int_0^1\tilde{\psi_i}(x)\phi_j(x)dx=0,\\ &\qquad i,j=0,1,...,m.\end{aligned} \tag{2.14}$$

Now we can introduce the $\tilde{\phi_i}(x)$ and $\tilde{\psi_i}(x)$ as polynomials and piecewise polynomials of degree m respectively, by

$$\tilde{\phi_i}(x)=\begin{cases}a_{i1}+a_{i2}x+...+a_{i,m+1}x^m, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases} \tag{2.15}$$

$$\tilde{\psi_i}(x)=\begin{cases}b^1_{i1}+b^1_{i2}x+...+b^1_{i,m+1}x^m, & 0\le x<\frac{1}{2},\\ b^2_{i1}+b^2_{i2}x+...+b^2_{i,m+1}x^m, & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise}.\end{cases} \tag{2.16}$$

Based on biorthogonal conditions (2.14), we can show that coefficient $a_{i,j}$, $b^1_{i,j}$ and $b^2_{i,j}, i=0,...,m$ and $j=1,...,m+1$, are uniquely determined (: see [9]). For example, we compute the dual multiwavelets corresponding to

(2.9) and (2.11) as

$$\tilde{\phi_0}(x)=\begin{cases} 3-4x, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}\quad,\qquad \tilde{\phi_1}(x)=\begin{cases} -1+4x, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}$$
$$\tilde{\psi_0}(x)=\begin{cases} 2\sqrt{2}(1-4x), & 0\le x<\frac{1}{2},\\ -2\sqrt{2}(3-4x), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise},\end{cases}\quad,\qquad \tilde{\psi_1}(x)=\begin{cases} \sqrt{10}(1-4x), & 0\le x<\frac{1}{2},\\ \sqrt{10}(3-4x), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise}.\end{cases} \tag{2.17}$$

Also computation of dual multiwavelets corresponding to Eq. (2.12), yields

$$\begin{aligned}
\tilde{\phi_0}(x)&=\begin{cases} \frac{11}{2}-18x+\frac{27}{2}x^2, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}\\
\tilde{\phi_1}(x)&=\begin{cases} \frac{-7}{2}-27x+27x^2, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}\\
\tilde{\phi_2}(x)&=\begin{cases} 1-9x+\frac{27}{2}x^2, & 0\le x<1,\\ 0, & \text{otherwise},\end{cases}\\
\tilde{\psi_0}(x)&=\begin{cases} \sqrt{10}(\frac{7}{4}-\frac{33}{2}x+27x^2), & 0\le x<\frac{1}{2},\\ -\sqrt{10}(\frac{49}{4}-\frac{75}{2}x+27x^2), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise},\end{cases}\\
\tilde{\psi_1}(x)&=\begin{cases} \sqrt{14}(\frac{9}{4}-\frac{45}{2}x+\frac{81}{2}x^2), & 0\le x<\frac{1}{2},\\ \sqrt{14}(\frac{81}{4}-\frac{117}{2}x+\frac{81}{2}x^2), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise},\end{cases}\\
\tilde{\psi_3}(x)&=\begin{cases} -\sqrt{14}(1-12x+27x^2), & 0\le x<\frac{1}{2},\\ \sqrt{14}(16-42x+27x^2), & \frac{1}{2}\le x<1,\\ 0, & \text{otherwise}.\end{cases}
\end{aligned} \tag{2.18}$$

Now, suppose $\Lambda(x)$ and $\tilde{\Lambda}(x)$ are two vector functions as

$$\Lambda(x)=\begin{bmatrix}\phi_0(x)\\ \vdots\\ \phi_m(x)\\ \psi_0(x)\\ \vdots\\ \psi_i(2^lx-k)\\ \vdots\\ \psi_m(2^Jx-2^J+1)\end{bmatrix},\quad \tilde{\Lambda}(x)=\begin{bmatrix}\tilde{\phi_0}(x)\\ \vdots\\ \tilde{\phi_m}(x)\\ \tilde{\psi_0}(x)\\ \vdots\\ \tilde{\psi_i}(2^lx-k)\\ \vdots\\ \tilde{\psi_m}(2^Jx-2^J+1)\end{bmatrix}. \tag{2.19}$$

So, we can approximate a function $f(x)$ defined on $[0,1]$ by the Flatlet multiwavelets [14] as

$$f(x)\simeq\Lambda^T(x).\tilde{\mathbf{C}}, \tag{2.20}$$

or

$$f(x)\simeq\tilde{\Lambda}^T(x).\tilde{\mathbf{C}}, \tag{2.21}$$

where $\mathbf{C}$ and $\tilde{\mathbf{C}}$ are N-vectors as

$$\begin{aligned}
\mathbf{C}&=[c_0,...,c_m,d_{0,0,0},...,d_{i,l,k},...,d_{m,J,2^J-1}],\\
\tilde{\mathbf{C}}&=[\tilde{c}_0,...,\tilde{c}_m,\tilde{d}_{0,0,0},...,\tilde{d}_{i,l,k},...,\tilde{d}_{m,J,2^J-1}],
\end{aligned}$$

in which $N=2^J(m+1)$.

Now, we can rewrite Eqs. (2.20) and (2.21) respectively as

$$f(x) \simeq \sum_{i=0}^{m} c_i \phi_i(x) + \sum_{i=0}^{m} \sum_{l=0}^{J} \sum_{k=0}^{2^j-1} d_{i,l,k} \psi_{i,l,k}(x), \tag{2.22}$$

and

$$f(x) \simeq \sum_{i=0}^{m} \tilde{c}_i \tilde{\phi}_i(x) + \sum_{i=0}^{m} \sum_{l=0}^{J} \sum_{k=0}^{2^J-1} \tilde{d}_{i,l,k} \tilde{\psi}_{i,l,k}(x), \tag{2.23}$$

where $\psi_{i,l,k}(x) = \psi_i(2^l x - k)$ and $\tilde{\psi}_{i,l,k}(x) = \tilde{\psi}_i(2^l x - k)$. The vectors of $\mathbf{C}$ and $\tilde{\mathbf{C}}$ can be obtained respectively as

$$\mathbf{C} = \int_0^1 f(x).\tilde{\Lambda}^T(x)dx, \tag{2.24}$$

and

$$\tilde{\mathbf{C}} = \int_0^1 f(x).\Lambda^T(x)dx. \tag{2.25}$$

In other words

$$c_i = \int_0^1 f(x).\tilde{\phi}_i(x)dx \quad , \quad d_{i,l,k} = \int_0^1 f(x).\tilde{\psi}_{i,j,k}(x)dx \tag{2.26}$$

and

$$\tilde{c}_i = \int_0^1 f(x).\phi_i(x)dx \quad , \quad \tilde{d}_{i,l,k} = \int_0^1 f(x).\psi_{i,j,k}(x)dx. \tag{2.27}$$

It is useful to note that the dual Flatlet multiwavelets are defined based on polynomials and have more flexibility in approximating functions so we use the last relation and Eq. (2.21) because of the higher order of accuracy.

3. Description of Flatlet Oblique Multiwavelet Method for Solving Fractional Volterra Integro-Differential Equation

As we mentioned in section 1, the purpose of this paper is to provide a numerical solution for the linear fractional Volterra integro-differential equation as

$$D^\alpha y(x) + q(x)y(x) = g(x) + \lambda \int_0^x K(x,t)y(t)dt, \tag{3.1}$$

with initial conditions $y^j(0) = c_j, j = 0, 1, ..., n-1$ and $n-1 < \alpha < n$. Also, D^α represents fractional derivative of the order $\alpha > 0$ of Caputo sense, $n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$. The continuous functions $q(x)$, $g(x)$, and $K(x,t)$ are given and $y(x)$ is the unknown function to be determined.

3.1. Constructing the Operational matrices.

In order to solve this problem, we must construct the fractional derivative of Caputo sense and the Volterra integral operational matrices. For this purpose, first we approximate the unknown function $y(x)$ by scaling and mother wavelet functions of dual flatlet multiwavelets as

$$y(x) = C^T \tilde{\Lambda}(x), \tag{3.2}$$

where $\mathbf{C}$ is an unknown vector of order $(m+1)2^J$ as follows

$$\mathbf{C} = [c_1, c_2, ..., c_{(m+1)2^J}]^T.$$

By fractional derivation both sides of equation (3.2) the relation

$$D^\alpha y(x) = C^T D^\alpha \tilde{\Lambda}(x) = C^T \mathbf{D} \tilde{\Lambda}(x), \tag{3.3}$$

is obtained. Where $\mathbf{D}$ is fractional derivation operational matrix that is calculated from the flatlet multiwavelet properties described in Eq. (2.14) as

$$\mathbf{D}=\int_0^1 \left(D^{\alpha}\tilde{\Lambda}(x)\right).\Lambda^T(x)dx. \tag{3.4}$$

More precisely, using the definition of fractional derivation of Caputo sens, the entries of matrix D are calculated as

$$\mathbf{D}_{i,j}(x)=\int_0^1 \left(\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{\tilde{\Lambda}_i^{(n)}(t)}{(x-t)^{\alpha+1-n}}dt\right)\Lambda_j(x)dx, \quad n-1<\alpha<n. \tag{3.5}$$

In other words, the fractional derivative was taken from all the basic functions in the vector $\tilde{\Lambda}(x)$. The expansion of these fractional derivatives was written in terms of the basic functions of the Flatlet multiwavelet and the coefficients of the expansion of the fractional derivative were calculated using the property of orthogonality between the initial wavelet and their duals. These obtained coefficients made the entries of the operational matrix $\mathbf{D}$

Also, in order to calculate the Volterra integral operational matrix, assume

$$\int_0^x y(t)dt=\int_0^x C^T\tilde{\Lambda}(t)dt=C^T\int_0^x\tilde{\Lambda}(t)dt=C^T\mathbf{P}\tilde{\Lambda}(x), \tag{3.6}$$

where $\mathbf{P}$ is the Volterra integral operational matrix whose entries is calculated using the flatlet multiwavelet properties described in Eq. (2.14) as

$$\mathbf{P}_{i,j}(x)=\int_0^1\left(\int_0^x\tilde{\Lambda}_i(t)dt\right)\Lambda_j(x)dx \tag{3.7}$$

In a similar way, the Volterra integral was taken from all the basic functions in the vector $\tilde{\Lambda}(x)$. The expansion of these functions was written in terms of the basic functions of the Flatlet multiwavelet and the coefficients of the expansion of the Volterra Integral were calculated using the property of orthogonality between the initial wavelet and their duals. These obtained coefficients made the entries of the operational matrix $\mathbf{P}$.

3.2. **Solving Fractional Volterra Integro-Differential Equation.** After calculating the operational matrices needed to solve Eq. (3.1), we extand the functions $y(x)$, $q(x)$, $g(x)$ and $K(x,t)$ by flatlet oblique multiwavelet basis respectively as

$$y(x)=C^T\tilde{\Lambda}(x), \tag{3.8}$$

$$q(x)=Q^T\tilde{\Lambda}(x), \tag{3.9}$$

$$g(x)=G^T\tilde{\Lambda}(x), \tag{3.10}$$

and

$$K(x,t)=\tilde{\Lambda}^T(t)K^T\tilde{\Lambda}(x). \tag{3.11}$$

Using Eq. (2.25), the coefficient vectors Q and G and the coefficient matrix K are determined respectively as

$$Q=\int_0^1 q(x).\Lambda^T(x)dx, \tag{3.12}$$

$$G=\int_0^1 g(x).\Lambda^T(x)dt, \tag{3.13}$$

and

$$K=\int_0^1\left(\int_0^1 K(x,t).\Lambda(t)dt\right).\Lambda(x)dx. \tag{3.14}$$

Where C is unknown vector to be determined. By substituting the operational matrix $\mathbf{D}$ whose entries calculated in Eq. (3.5) and Eqs. (3.8), (3.10) and (3.11) in Eq. (3.1) we can write

$$C^T.\mathbf{D}.\tilde{\Lambda}(x)+Q^T.\tilde{\Lambda}(x).\tilde{\Lambda}^T(x).C=G^T.\tilde{\Lambda}(x)+\int_0^x\tilde{\Lambda}^T(x).K^T.\tilde{\Lambda}(t).\tilde{\Lambda}^T(t).Cdt. \tag{3.15}$$

Let $H_N(t)$ be a square matrix of dimensions $(m+1)2^J \times (m+1)2^J$ as

$$H_N(t) = \tilde{\Lambda}(t).\tilde{\Lambda}^T(t), \tag{3.16}$$

whose entries are obtained by

$$H_{N\,i,j}(t) = \tilde{\Lambda}_i(t).\tilde{\Lambda}_j(t). \tag{3.17}$$

So, the Eq. (3.15) can be written as

$$C^T.\mathbf{D}.\tilde{\Lambda}(x) + Q^T.H_N(x).C = G^T.\tilde{\Lambda}(x) + \int_0^x \tilde{\Lambda}^T(x).K^T.H_N(t).Cdt. \tag{3.18}$$

If we define the square matrix $\mathbf{B}$ of dimension $(m+1)2^J \times (m+1)2^J$ as

$$H_N(t).C = \mathbf{B}_N.\tilde{\Lambda}(t), \tag{3.19}$$

then, using the Flatlet multiwavelets biorthogonality properties (2.14) we can write

$$\int_0^1 H_N(t).C.\Lambda^T(t)dt = \int_0^1 \mathbf{B}_N.\tilde{\Lambda}(t).\Lambda^T(t)dt, \tag{3.20}$$

and $\mathbf{B}_N$ is calculated as

$$\mathbf{B}_N = \int_0^1 H_N(t).C.\Lambda^T(t)dt. \tag{3.21}$$

Substituting Eq. (3.21) in Eq. (3.18) gives the following result

$$\begin{aligned} C^T.\mathbf{D}.\tilde{\Lambda}(x) + Q^T.\mathbf{B}_N.\tilde{\Lambda}(x) &= G^T.\tilde{\Lambda}(x) + \lambda.\int_0^x \tilde{\Lambda}^T(x).K^T.\mathbf{B}_N.\tilde{\Lambda}(t)dt \\ &= G^T.\tilde{\Lambda}(x) + \lambda.\tilde{\Lambda}^T(x).K^T.\mathbf{B}_N.\int_0^x \tilde{\Lambda}(t)dt \\ &= G^T.\tilde{\Lambda}(x) + \lambda.\tilde{\Lambda}^T(x).K^T.\mathbf{B}_N.\mathbf{P}.\tilde{\Lambda}(x), \end{aligned} \tag{3.22}$$

where $\mathbf{P}$ is operational matrix of Voltera integration.

In order to approximate the solution of Eq. (3.1), we use $(m+1)2^J$ collocation points. First, set $N = (m+1)2^J$, so the suitable collocation points are Newoton-Cotes nodes as

$$x_i = \frac{2i-1}{2N}, \quad i = 1, 2, ..., N. \tag{3.23}$$

Suppose the matrix L_N is a square matrix of dimension $N \times N$ whose columns are the value of $\tilde{\Lambda}(x)$ at the above points as

$$L_N = [\tilde{\Lambda}(x_1), \tilde{\Lambda}(x_2), ..., \tilde{\Lambda}(x_N)]. \tag{3.24}$$

Furthermore,

$$\tilde{\Lambda}(x_i) = L_N.e_i, \tag{3.25}$$

where e_i is $N \times 1$ vector that entire i is 1 and the others are 0.

Now, by applying collocation points and Eq. (3.25), Eq. (3.22) can be rewritten as

$$C^T.\mathbf{D}.L_N.e_i + Q^T.\mathbf{B}_N.L_N.e_i = G^T.L_N.e_i + \lambda.{e_i}^T.{L_N}^T.K^T.\mathbf{B}_N.\mathbf{P}.L_N.e_i. \tag{3.26}$$

Eq. (3.26) makes a set of algebraic equations with $(m+1)2^J$ equations and $(m+1)2^J$ unknowns $\{c_1, c_2, ..., c_N\}$ that can be easily solved.

4. Error Analysis and Convergence of the Method

In this section, we discusse the error analysis and convergence of the presented method.

Theorem 4.1. *For Flatlet oblique multiwavelet functions ψ_i of order m and $i = 0, 1, ..., m$, which have $m + i + 1$ vanishing moments from Eq. (2.6), if $f(x) \in \mathbf{C}^{(\sum_{i=0}^{m} m+i+1)}(\mathbb{R})$ then*

$$\left|\tilde{d}_{i,j,k}\right| \leq C.2^{-(\sum_{i=0}^{m} m+i+1)} \max_{\zeta\in[0,1]} |f^p(\zeta)|, \tag{4.1}$$

in which C is a constant independent of j and f.

Proof. For each $x \in [0, 1]$, the Taylor expansion of function $f(x)$ at the point $x = \frac{k}{2^j}$ is as follows

$$f(x) = \left(\sum_{l=0}^{(m+1)} f^{(l)}(\frac{k}{2^j})\frac{(x-\frac{k}{2^j})^l}{l!}\right) + f^{(m+i+1)}(\zeta)\frac{(x-\frac{k}{2^j})^{\sum_{i=0}^{m} m+i+1}}{(m+i+1)!}, \tag{4.2}$$

In which $\zeta \in [\frac{k}{2^j}, x]$. By substituting Eq. (4.2) in Eq. (2.27) we can write

$$\begin{aligned}
\tilde{d}_{i,j,k} &= \int_0^1 f(x).\psi_{i,j,k}(x)dx \\
&= \sum_{l=0}^{m+i} f^{(l)}(\frac{k}{2^j})\frac{1}{l!}\int_0^1 (x-\frac{k}{2^j})^l.\psi_{i,j,k}(x)dx \\
&+ \frac{1}{(\sum_{i=0}^{m} m+i+11)!}\int_0^1 f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)(x-\frac{k}{2^j})^{\sum_{i=0}^{m} m+i+1}.\psi_{i,j,k}(x)dx.
\end{aligned} \tag{4.3}$$

Suppose that $t = 2^j x - k$ so

$$\begin{aligned}
\int_0^1 (x-\frac{k}{2^j})^l \psi_{i,j,k}(x)dx &= \int_0^1 (x-\frac{k}{2^j})^l \psi_i(2^j x-k)dx \\
&= \int_0^1 \left(\frac{t}{2^j}\right)^l \psi_i(t).2^{-j}dt \\
&= 2^{-j(l+1)}\int_0^1 t^l\psi_i(t)dt \quad , \quad l = 0, 1, ..., m+i.
\end{aligned} \tag{4.4}$$

Now, using vanishing moments property we will have

$$\int_0^1 (x-\frac{k}{2^j})^l \psi_{i,j,k}(x)dx = 0 \quad , \quad l = 0, 1, ..., m+i, \tag{4.5}$$

and Eq. (4.3) can be written as

$$\begin{aligned}
\left|\tilde{d}_{i,j,k}\right| &= \frac{1}{(\sum_{i=0}^{m} m+i+1)!}\left|\int_0^1 f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)(x-\frac{k}{2^j})^{\sum_{i=0}^{m} m+i+1}\psi_i(2^j x-k)dx\right| \\
&\leq \frac{1}{(\sum_{i=0}^{m} m+i+1)!}\max_{\zeta\in[0,1]}\left|f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)\right|\int_0^1\left|(x-\frac{k}{2^j})^{m+i+1}\psi_i(2^j x-k)\right|dx \\
&= 2^{-((\sum_{i=0}^{m} m+i+1)+1)j}\frac{1}{(\sum_{i=0}^{m} m+i+1)!}\max_{\zeta\in[0,1]}\left|f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)\right|\int_0^1\left|t^{\sum_{i=0}^{m} m+i+1}\psi_i(t)\right|dt,
\end{aligned} \tag{4.6}$$

by assuming

$$C = \frac{1}{(\sum_{i=0}^{m} m+i+1)!}\int_0^1\left|t^{\sum_{i=0}^{m} m+i+1}\psi_i(t)\right|dt, \tag{4.7}$$

the proof ends. $\square$

Theorem 4.2. *For Flatlet oblique multiwavelet functions ψ_i of order m and $i = 0, 1, ..., m$, with compact support which have $N_i = \sum_{i=0}^{m} m + i + 1$ vanishing moments we can say*

$$|\epsilon_J(x)| = \mathbf{O}\left(2^{-(\sum_{i=0}^{m} m+i+1)J}\right). \tag{4.8}$$

Proof. Since

$$f(x) = \sum_{i=0}^{m} \tilde{c}_i\tilde{\phi}_i(x) + \sum_{i=0}^{m}\sum_{j=0}^{J-1}\sum_{k=0}^{2^j-1} \tilde{d}_{i,j,k}\tilde{\psi}_i(2^j x - k), \tag{4.9}$$

therefore, the error of this approximation will be as follows

$$\epsilon_J(x) = \sum_{J}^{\infty}\sum_{k=0}^{2^j-1} \tilde{d}_{i,j,k}\tilde{\psi}_{i,j,k}(x). \tag{4.10}$$

Suppose

$$C_{\tilde{\psi}} = \max_{x\in[0,1]}\left|\tilde{\psi}_i(2^j x - k)\right| = \max_{t\in[0,1]}\left|\tilde{\psi}_i(t)\right|, \tag{4.11}$$

using (4.11) we can write

$$\left|\tilde{d}_{i,j,k}\tilde{\psi}_{i,j,k}(x)\right| \le C2^{-(((\sum_{i=0}^{m} m+i+1))+1)j} \max_{\zeta\in[0,1]}\left|f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)\right| C_{\tilde{\psi}}. \tag{4.12}$$

Now, we obtain

$$\begin{aligned}\sum_{k=0}^{2^j-1}\left|\tilde{d}_{i,j,k}\tilde{\psi}_{i,j,k}(x)\right| &\le C_{\tilde{\psi}}C2^{-((\sum_{i=0}^{m} m+i+1)+1)j}2^j \max_{\zeta\in[0,1]}\left|f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)\right| \\ &= C_{\tilde{\psi}}C2^{-(\sum_{i=0}^{m} m+i+1)j} \max_{\zeta\in[0,1]}\left|f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)\right|,\end{aligned} \tag{4.13}$$

substituting (4.13) in (4.10) we can write

$$\begin{aligned}|\epsilon_J(x)| &\le C_{\tilde{\psi}}C \max_{\zeta\in[0,1]}\left|f^{(\sum_{i=0}^{m} m+i+1)}(\zeta)\right|\sum_{j=J}^{\infty} 2^{-(\sum_{i=0}^{m} m+i+1)j} \\ &= C_{\tilde{\psi}}C \max_{\zeta\in[0,1]}\left|f^{(m+i+1)}(\zeta)\right| \frac{2^{-(m+i+1)J}}{1-2^{-(m+i+1)}}.\end{aligned} \tag{4.14}$$

Therefore, we conclude that for any desired x, the approximation error will be as follows

$$|\epsilon_J(x)| = \mathbf{O}\left(2^{-(\sum_{i=0}^{m} m+i+1)J}\right), \tag{4.15}$$

and as m and J increase, the error decreases. ☐

Lemma 4.3. *[27] Let $X = L^2([0,1])$, indicates the vector space of square-summable functions defined on $[0,1]$ and Ω be a Volterra integral operator on X defined by*

$$\Omega(u(x)) = \int_0^x \kappa(x,t)u(t)dt, \qquad \forall u \in X, \tag{4.16}$$

and for the kernel $\kappa(x,t)$ we have

$$\int_0^1\int_0^1 |\kappa(x,t)|dtdx = \rho^2, \tag{4.17}$$

or $\sup_{x,t}\kappa(x,t) = \rho$ and ρ is a constant. Then Ω is bounded in $L^2([0,1])$. That is,

$$||\Omega(u(x))||_2 \le \rho||u||_2 \tag{4.18}$$

Lemma 4.4. *Let $u(x)$ be sufficiently smooth function in $L^2([0,1])$ and $D^{\alpha}(x)$ indicates the approximation of Caputo fractional derivative of the order $\alpha > 0$ of $u(x)$, $D^{\alpha}u(x)$. Assuming that $D^{\alpha}u(x)$ is bounded by a constant ξ, $D^{\alpha}u(x) \leq \xi$, in this case, there exists an integer like γ such that*

$$||D^{\alpha}u(x) - D^{\alpha}u_N(x)||_2{}^2 \leq \xi^2.\gamma. \tag{4.19}$$

Proof. Let

$$D^{\alpha}u(x) = \sum_{i=1}^{\infty} c_i \tilde{\Lambda}_i(x), \tag{4.20}$$

now, if we consider the first $N = (m+1)2^J$ terms of the sum above, we will have

$$D^{\alpha}u_N(x) = \sum_{i=1}^{N} c_i \tilde{\Lambda}_i(x), \tag{4.21}$$

and Eqs. (4.20) and (4.21) yields

$$D^{\alpha}u(x) - D^{\alpha}u_N(x) = \sum_{i=N+1}^{\infty} c_i \tilde{\Lambda}_i(x). \tag{4.22}$$

Using Eq. (2.19), since the end entries of the vector $\tilde{\Lambda}(x)$ are composed of $\psi_i(x)$ and $i = 1, ..., N$, it can be written as

$$\begin{aligned} ||D^{\alpha}u(x) - D^{\alpha}u_N(x)||_2{}^2 &= \int_0^1 \left(D^{\alpha}u(x) - D^{\alpha}u_N(x)\right)^2 dx \\ &= \int_0^1 \left(\sum_{i=N+1}^{\infty} c_i \tilde{\Lambda}_i(x)\right)^2 dx \\ &= \sum_{i=N+1}^{\infty} c_i{}^2 \begin{cases} \int_0^{\frac{1}{2}} \left(s_{i1}^1 + s_{i2}^1 x + ... + s_{i,2m+1}^1 x^{2m}\right) dx, & 0 \leq x < \frac{1}{2} \\ \int_{\frac{1}{2}}^1 \left(s_{i1}^2 + s_{i2}^2 x + ... + s_{i,2m+1}^2 x^{2m}\right) dx, & \frac{1}{2} \leq x < 1 \\ 0, & \text{otherwise} \end{cases} \\ &+ \sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} c_i.c_j \begin{cases} \int_0^{\frac{1}{2}} \left(p_{i1}^1 + p_{i2}^1 x + ... + p_{i,2m+1}^1 x^{2m}\right) dx, & 0 \leq x < \frac{1}{2} \\ \int_{\frac{1}{2}}^1 \left(p_{i1}^2 + p_{i2}^2 x + ... + p_{i,2m+1}^2 x^{2m}\right) dx, & \frac{1}{2} \leq x < 1 \\ 0, & \text{otherwise} \end{cases} \\ &= \sum_{i=N+1}^{\infty} c_i{}^2.\mathbf{M} + \sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} c_i.c_j.\mathbf{P}, \end{aligned} \tag{4.23}$$

where

$$\mathbf{M} = \begin{cases} \frac{1}{2}.s_{i1}^1 + \frac{1}{8}.s_{i2}^1 + ... + \frac{(\frac{1}{2})^{2m+1}}{2m+1}.s_{i,2m+1}^1, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}.s_{i1}^2 + \frac{1}{8}.s_{i2}^2 + ... + \frac{(\frac{1}{2})^{2m+1}}{2m+1}.s_{i,2m+1}^2, & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathbf{P} = \begin{cases} \frac{1}{2}.p_{i1}^1 + \frac{1}{8}.p_{i2}^1 + ... + \frac{(\frac{1}{2})^{2m+1}}{2m+1}.p_{i,2m+1}^1, & 0 \leq x < \frac{1}{2}, \\ \frac{1}{2}.p_{i1}^2 + \frac{1}{8}.p_{i2}^2 + ... + \frac{(\frac{1}{2})^{2m+1}}{2m+1}.p_{i,2m+1}^2, & \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise}, \end{cases}$$

also

$$s_{ik} = \sum_{i=N+1}^{\infty} \sum_{k=1}^{2m+1} b_{ik}.b_{ki} \quad , \quad p_{ik} = \sum_{i=N+1}^{\infty} \sum_{k=1}^{2m+1} \sum_{j=N+1}^{\infty} b_{ik}.b_{kj}. \tag{4.24}$$

On the other hand, according to the Eq. (2.27), $c_i = \int_0^1 D^{\alpha}u(x).\Lambda_i(x)dx$, so from Eq. (4.7) we can write

$$c_i \leq \xi \int_0^1 \Lambda_i(x)dx = \xi.C,$$

and as a result

$$|c_i|^2 \leq (\xi.C)^2, \tag{4.25}$$

where

$$C = \frac{1}{(m+i+1)!} \int_0^1 \left|t^{m+i+1}\psi_i(t)\right| dt. \tag{4.26}$$

So

$$\begin{aligned} ||D^{\alpha}u(x) - D^{\alpha}u_N(x)||_2{}^2 &= \sum_{i=N+1}^{\infty} c_i{}^2.\mathbf{M} + \sum_{i=N+1}^{\infty} \sum_{j=N+1}^{\infty} c_i.c_j.\mathbf{P} \\ &\leq \sum_{i=N+1}^{\infty} (\xi.C)^2 (\mathbf{M}+\mathbf{P}) = \xi^2.\gamma, \end{aligned} \tag{4.27}$$

and $\gamma = C^2(\mathbf{M}+\mathbf{P})$. □

Theorem 4.5. *Let us consider all assumptions of theorems (4.1) and (4.2) and lemmas (4.3) and (4.4), also let $y_N(x)$ be the approximate solution of fractional Volterra integro-differential Eq. (3.1) given by*

$$y_N(x) = \sum_{i=1}^{N} c_i \tilde{\Lambda}_i(x), \quad N = (m+1)2^J,$$

and satisfy the initial conditions $y^j(0) = c_j, j = 0, 1, ..., n-1$ and $n-1 < \alpha < n$, then we have $||y(x) - y_N(x)|| \to 0$ when $N \to \infty$.

Proof. Let $\epsilon_N(x) = y(x) - y_N(x)$ represent the error function of the approximate solution $y_N(x)$ for the exact solution $y(x)$ and $N = (m+1)2^J$. By substituting $y_N(x)$ in the Eq. (3.1) we get

$$D^{\alpha}y_N(x) + q(x)y_N(x) = g(x) + \lambda \int_0^x K(x,t)y_N(t)dt, \tag{4.28}$$

and also we can write

$$D^{\alpha}y(x) - D^{\alpha}y_N(x) + q(x)\left(y(x) - y_N(x)\right) = \lambda \int_0^x K(x,t)\left(y(t) - y_N(t)\right)dt. \tag{4.29}$$

So

$$\begin{aligned} q(x)\left(\epsilon_N(x)\right) &= \int_0^x K(x,t)\epsilon_N(t)dt - \left(D^{\alpha}y(x) - D^{\alpha}y_N(x)\right) \\ &\leq \left|\int_0^x K(x,t)\epsilon_N(t)dt\right| + \left|D^{\alpha}y(x) - D^{\alpha}y_N(x)\right| \\ &\leq \rho \left|\int_0^x \epsilon_N(t)dt\right| + \left|D^{\alpha}y(x) - D^{\alpha}y_N(x)\right|. \end{aligned} \tag{4.30}$$

Now, by Eq. (4.29) and Gronwall's inequality [28], we can write

$$||q(x)\left(\epsilon_N(x)\right)||_2 \leq ||D^{\alpha}y(x) - D^{\alpha}y_N(x)||_2 \leq \xi\sqrt{\gamma}. \tag{4.31}$$

Using the proofs of theorem (4.1) and lemma (4.3), since $\gamma = (\mathbf{M}+\mathbf{P})\Big(\frac{1}{(m+i+1)!}\int_0^1 \big|t^{m+i+1}\psi_i(t)\big|\,dt\Big)^2$ and $q(x) \neq 0$, therefore $\epsilon_N(x) \to 0$ or $y(x) \to y_N(x)$ as $N \to \infty$. □

5. Numerical examples

In order to illustrate the performance of the proposed method and justify the accuracy and efficiently of the presented method, we consider the following examples.

Remark 5.1. First we consider the fractional Volterra integro-differential equation given by, [32]

$$D^{\frac{1}{2}}y(x) = y(x) + \frac{8}{3\Gamma(0.5)}x^{1.5} - x^2 - \frac{1}{3}x^3 + \int_0^x y(t)dt, \qquad (5.1)$$
$$y(0)=0, \quad y'(0)=0,$$

whith the exact solution $y(x) = x^2$. In Table 1 , the absolute error values for different m and J is presented. By looking at the table, it is clear that the exact answer is obtained even for low m and J. Also we compared our results with LWPGM method presented in [32]. Figure 1 shows the approximation processes in this example for $m = 4$ and $J = 1$. Also Figure.2 shows the errore rate for several m and J briefly.

Remark 5.2. Consider the following fractional Volterra integro-differential equation

$$D^{0.75}y(x) = (x\cos x - \sin x)y(x) + \frac{1}{\Gamma(1.25)}x^{0.25} + \int_0^x x\sin t y(t)dt, \qquad (5.2)$$
$$y(0)=0, \quad y'(0)=0.$$

The exact solution of above equation is $y(x) = x$. This problem has been solved by Flatlet oblique multiwavelet method and you can see the absolute values of errors for different values of m and J in Table 2. The approximation and error diagrams are shown in Figures 3 and 4 for different values of m and J respectively.

Remark 5.3. Consider the following fractional Volterra integro-differential equation

$$D^{\sqrt{3}}y(x) = \frac{2}{\Gamma(3-\sqrt{3})}x^{2-\sqrt{3}} + 2\sin x - 2x + \int_0^x \cos(x-t)y(t)dt, \qquad (5.3)$$
$$y(0)=0, \quad y'(0)=0.$$

with the exact solution $y(x) = x^2$. The absolute values of errors for different values of m and J is shown in Table 3. Also Figures 5 and 6 represent the approximation and error diagrams for different values of m and J.

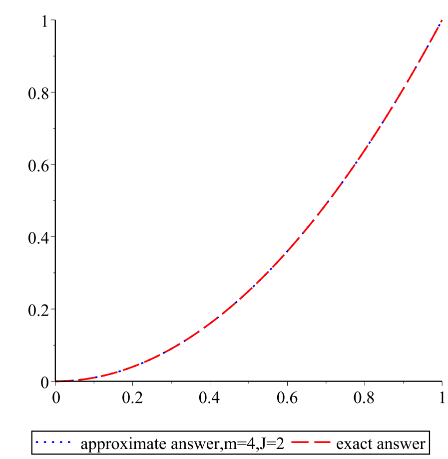

Figure 1. The numerical solution of Example 5.1 for $m = 4$ and $J = 2$ with the exact answer.

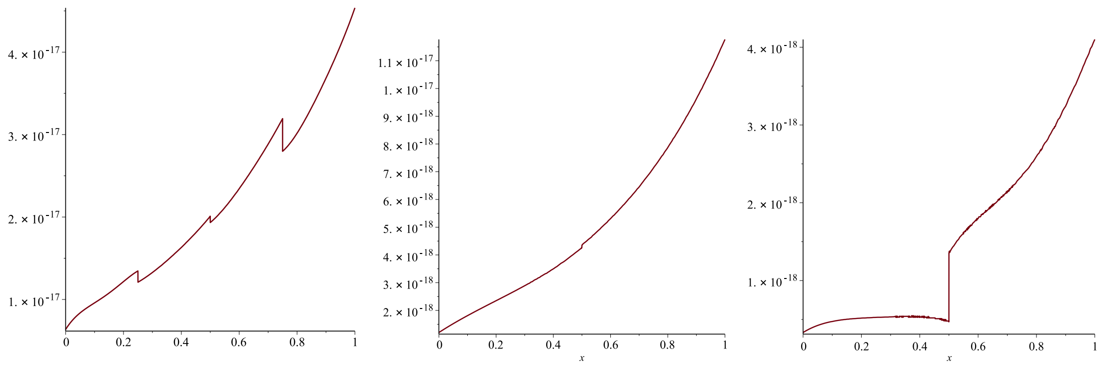

FIGURE 2. The error value diagrams of Example 5.1, the left one for $m = 3$ and $J = 1$, the middle one for $m = 4$ and $J = 1$ and the right one for $m = 4$ and $J = 2$.

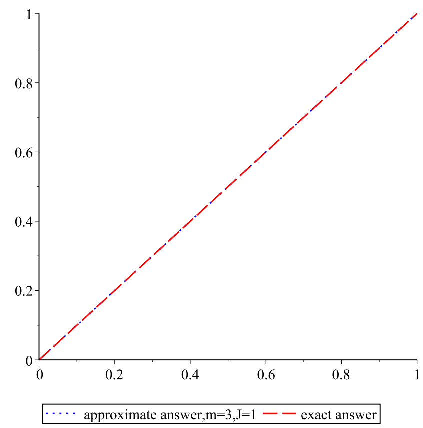

FIGURE 3. The numerical solution of Example 5.2 for $m = 4$ and $J = 1$ with the exact answer.

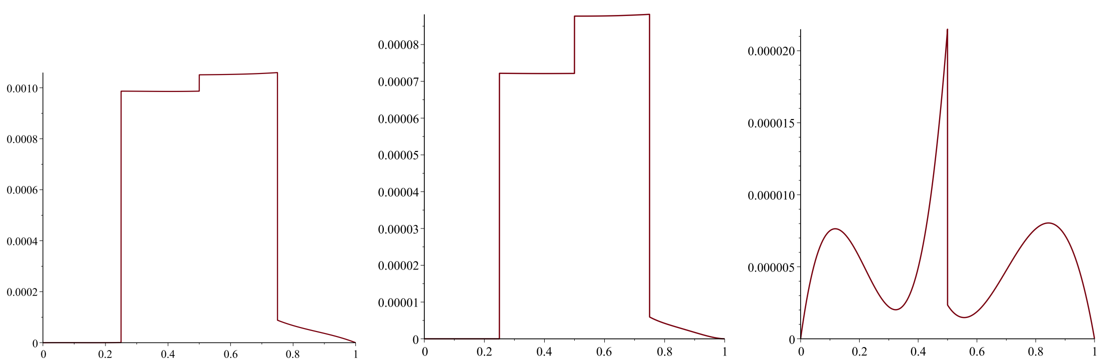

FIGURE 4. The error value diagrams of Example 5.2, the left one for $m = 3$ and $J = 1$, the middle one for $m = 3$ and $J = 2$ and the right one for $m = 4$ and $J = 1$.

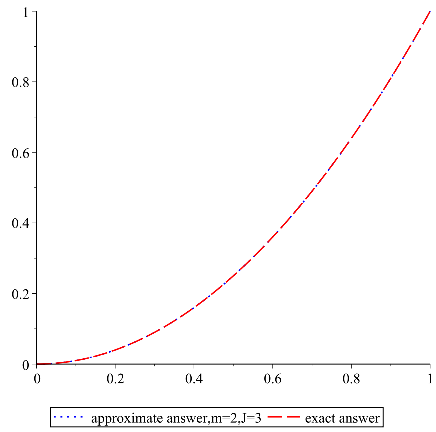

FIGURE 5. The numerical solution of Example 5.3 for $m = 2$ and $J = 3$ with the exact answer.

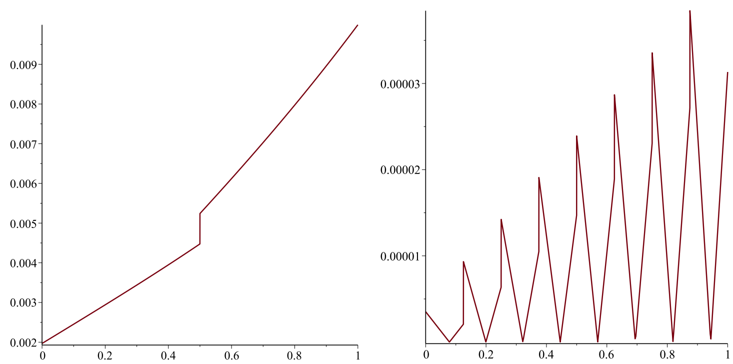

FIGURE 6. The error value diagrams of Example 5.3, the left one for $m = 3$ and $J = 1$ and $J = 2$ and the tight one for $m = 2$ and $J = 3$.

TABLE 1. Absolute values of errors for Example 5.1.

x	Exact	$m = 2$, $J = 1$	$m = 4$,$J = 1$	Errors by LWPGM[32]
0.0	0.0	1.83×10^{-19}	2.18×10^{-19}	3.88×10^{-16}
0.1	0.01	2.23×10^{-19}	3.32×10^{-19}	5.55×10^{-16}
0.2	0.04	2.77×10^{-19}	4.42×10^{-19}	6.66×10^{-16}
0.3	0.09	3.45×10^{-19}	5.52×10^{-20}	9.15×10^{-16}
0.4	0.16	4.20×10^{-19}	6.70×10^{-20}	1.27×10^{-15}
0.5	0.25	5.10×10^{-19}	8.10×10^{-19}	1.63×10^{-15}
0.6	0.36	5.30×10^{-19}	1.50×10^{-18}	2.04×10^{-15}
0.7	0.49	6.50×10^{-19}	1.79×10^{-18}	2.52×10^{-15}
0.8	0.64	8.10×10^{-19}	2.04×10^{-18}	3.27×10^{-15}
0.9	0.81	1.01×10^{-18}	2.34×10^{-18}	3.77×10^{-15}
1.0	1.0	1.20×10^{-18}	2.76×10^{-18}	4.21×10^{-15}

TABLE 2. Absolute values of errors for Example 5.2.

x	Exact	$m=3$, $J=1$	$m=4$,$J=1$	$m=4$,$J=2$
0.0	0.0	7.60×10^{-20}	8.34×10^{-20}	2.76×10^{-22}
0.1	0.1	7.50×10^{-6}	1.23×10^{-7}	5.15×10^{-10}
0.2	0.2	5.66×10^{-6}	5.63×10^{-8}	7.10×10^{-10}
0.3	0.3	2.22×10^{-6}	6.61×10^{-8}	7.21×10^{-6}
0.4	0.4	4.92×10^{-6}	7.71×10^{-8}	7.21×10^{-7}
0.5	0.5	2.15×10^{-5}	3.30×10^{-7}	7.22×10^{-7}
0.6	0.6	1.87×10^{-6}	2.60×10^{-6}	8.27×10^{-7}
0.7	0.7	4.75×10^{-6}	2.39×10^{-6}	8.80×10^{-7}
0.8	0.8	7.63×10^{-6}	2.41×10^{-6}	4.21×10^{-7}
0.9	0.9	7.16×10^{-6}	2.35×10^{-6}	1.75×10^{-7}
1.0	1.0	7.1×10^{-19}	1.60×10^{-18}	2.51×10^{-18}

TABLE 3. Absolute values of errors for Example 5.3.

x	Exact	$m=3$, $J=1$	$m=3$, $J=2$	$m=2$, $J=3$
0.0	0.0	1.96×10^{-3}	1.0×10^{-4}	3.52×10^{-6}
0.1	0.01	2.45×10^{-3}	1.5×10^{-4}	9.51×10^{-7}
0.2	0.04	2.94×10^{-3}	2.0×10^{-4}	8.25×10^{-8}
0.3	0.09	3.43×10^{-3}	3.1×10^{-4}	4.32×10^{-6}
0.4	0.16	3.94×10^{-3}	3.9×10^{-4}	1.23×10^{-5}
0.5	0.25	4.47×10^{-3}	4.8×10^{-4}	1.47×10^{-5}
0.6	0.36	6.12×10^{-3}	5.9×10^{-4}	1.03×10^{-5}
0.7	0.49	7.02×10^{-3}	7.1×10^{-4}	2.36×10^{-6}
0.8	0.64	7.97×10^{-3}	7.9×10^{-4}	9.26×10^{-6}
0.9	0.81	8.95×10^{-3}	9.4×10^{-4}	2.45×10^{-5}
1.0	1.0	9.99×10^{-3}	1.1×10^{-3}	3.13×10^{-5}

6. Conclusions

In this paper, the Flatlet oblique multiwavelets (FOM) was used to solve the fractional Volterra integro-differential equation. In this way, first by expanding the unknown function $y(x)$ by the dual functions of flatlet oblique multiwavelets and then by constructing the operator matrices related to fractional derivative and Volterra integration, we converted the fractional Volterra integro-differential equation into a set of algebraic equations and solved this system of algebraic equations using collocation points that are Newoton-Cotes nodes. Therefore, we obtained a suitable approximation for the solution of fractional Volterra integro-differential equation, the results of which are presented in different values of m and J. Also the convergence of method and error analysis were expressed as several theorems and their proofs. This method is computationally attractive and its practicality is demonstrated through illustrative examples. In this research, we have seen that by increasing the order of Flatlet multiwavelet (m) even using low-dimensional matrices, an acceptable approximation of the answer with fewer error values can be achieved.

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