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# A Numerical Method for Estimating the Dynamic Response of Structures

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**ABSTRACT:**

A novel numerical method is proposed for computing the seismic response of linear and nonlinear systems. Single-degree-of-freedom (SDOF) and multi-degree-of-freedom (MDOF) systems are covered. The method is called load impulse method (LIM) because it uses the load impulse concept in its formulation. LIM is first extended for analyzing linear damped systems whose damping ratios are almost greater than 1% and nonlinear systems in general. To formulate LIM, the governing differential equation of motion (DEOM) is modified to have appropriate form for numerical integration. Then, it is integrated over time step using trapezoidal integration rule. Rearranging the obtained equation, the required relations are generated for computing seismic response of dynamic systems through simple iteration. The seismic response of several linear and nonlinear structural systems under dynamic loads is determined through the proposed LIM. A detailed comparison is then carried out between the results of LIM and those obtained from Duhamel integral, Newmark-β, and Wilson-θ methods. The results clearly show that the proposed LIM can robustly estimate the displacement, velocity, and acceleration time-histories of the dynamic systems within satisfactory computational cost.

**KEYWORDS:**

Load Impulse Method (LIM), Numerical method, Dynamic response, Linear and nonlinear analyses, structural vibration, Newmark-β, Duhamel integral.

## 1. Introduction

Equation of motion for the single-degree-of-freedom (SDOF) systems is a fundamental issue in structural dynamics and vibration. The response of a structural system subjected to a time-dependent load function (e.g. earthquake load) is dominated by a ‘2nd-order’ differential equation (DE), which is called the governing differential equation of motion (DEOM). The dynamic response of structural systems is determined by solving this DE for successive time instance (Babaei, 2024). However, finding analytical solution is rarely possible in real cases and classic methods are almost impractical (Ebeling et al., 1997). Moreover, in most of structural cases, the seismic response of a multi-degree-of-freedom (MDOF) structure is estimated through analyzing an equivalent model (Chopra, 2003). As shown in the technical contexts (Vamvatsikos and Cornell, 2005), such a simplification decreases computational cost of MDOF structural analyses.

In the literature, there are numerous analysis methods for estimating the dynamic response of the buildings, e.g., Wilson-θ (Wilson, 1968), linear acceleration (Chung and Hulbert, 1993), Newmark-β (Newmark, 1959), HHT-α (Hilber et al., 1977), WBZ-α (Wood *et al*., 1980), the ρ-method (Bazzi and

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Anderheggen, 1982), HP-θ1 (Hoff and Pahl, 1988a; Hoff and Pahl, 1988b), Duhamel integral method (Clough and Penzien, 1995), and piecewise exact method (Veletsos et al., 1965; Chopra, 2012; Mohammadzadeh and Noh, 2014). In general, these methods are classified into two main groups. First, the procedures that are based on the superposition principle. These methods are limited to the linear systems and they estimate the dynamic response of structures by superposing numerous response contributions. Duhamel integral in time domain, Laplace, and Fourier transforms in frequency domain belong to this group (Hall et al., 2002; He, 2008; Rahmati et al., 2010). Second, the stepwise procedures which use direct integration scheme in their formulation. These methods are completely general to be applied to linear and nonlinear systems undergoing severely varying excitation functions. Euler-Gauss, Runge-Kutta, Newmark-β, and Wilson-θ methods are some of the most known integration-based methods in dynamics.

In recent decades, several simplified analysis methods have been extended, one of which is proposed by Li and Wu (2004) to determine the dynamic response of the inelastic SDOF systems with the time-varying mass and stiffness parameters. Wu (2013) proposed an iterative procedure to approximate the inelastic response of SDOF systems with the general nonlinear restoring forces. In another research, a new analytical method was proposed by the same authors for calculating the natural period of the SDOF systems. Serious attempts have been made by Chang (2004) to demonstrate the accuracy and efficiency of the Newmark-β method in solving the governing DE of nonlinear SDOF systems. Kazakov (2008) studied the response of SDOF systems using Duhamel integral method for some special dynamic load cases. The results showed that the Duhamel integral method is accurate enough in estimating the dynamic response of linear damped and undamped SDOF systems. A simple numerical method was then developed by Kurt and Çevik (2008) in which the Taylor polynomial was employed to estimate the dynamic response of the SDOF systems. Most recently, a novel single-step method is developed for analyzing dynamic systems (Kim, 2019). It is an explicit time integration method which is based on the Newmark approximation. A generalized semi-explicit method is also presented for solving dynamical problems of structures (Li et al., 2018). In an alternative work, two explicit methods, which benefit time integration scheme, are proposed based on displacement-velocity relations for treating the problems in structural dynamics (Zhang et al., 2019). Hanafi et al. (2024) improved some conventional methods to solve DE of motion. Recently, Atasoy et al. (2024), Babaei et al. (2021-2024) presented advanced techniques for time-history analysis of nonlinear systems. Although there exists a wide variety of methods, most are often complicated or need significant computational efforts.

Now, in this paper, a different methodology is introduced for estimating the dynamic response of SDOF and MDOF systems subjected to arbitrary loading function. It is inspired from dynamical concept of load impulse, called load impulse method (LIM). It is first formulated for linear SDOF systems. Then, the nonlinear version of LIM is proposed for SDOF systems. To be general, MDOF version of LIM is also discussed. Systematic algorithms are prepared for each formulation discussed in the context. Finally, the seismic response of several structural systems under dynamic loading is determined through LIM. A detailed comparison is then carried out between the results obtained from LIM, Duhamel integral, Newmark-β, and Wilson-θ methods. The results show that the proposed method is reliable and can satisfactorily estimate the dynamic response of linear and nonlinear damped SDOF as well as MDOF systems with reasonable computational costs. It is observed that, although the impulse technique may not be as fast as Newmark-β and Wilson-θ methods, it benefits very simple formulation which vastly facilitates computer programming of dynamic analysis.

## 2. Proposed procedure

Fig. 1 shows an idealized SDOF mass-spring system, which is a fundamental issue in the mechanical vibration. Denoting the relative displacement of mass by $x$ and its corresponding velocity and acceleration by $\dot{x}$ and $\ddot{x}$, respectively, the governing DEOM for an elastic SDOF system subjected to a time-varying external load of $F(t)$ can be presented by the following equation (Chopra, 2012):

$$kx + c\dot{x} + m\ddot{x} = F(t) \quad (1)$$

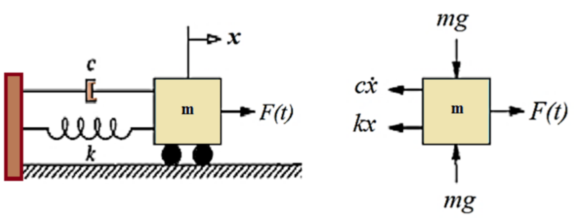


**Fig. 1**. An idealized SDOF mass-spring system subjected to $F(t)$ external load.

As proved in the literatures, the governing DEOM of a structure subjected to ground acceleration $\ddot{x}_g$ at the support, is written as:

$$kx + c\dot{x} + m\ddot{x} = -m\ddot{x}_g \quad (2)$$

where $m$, $c$, and $k$ are the mass, damping, and stiffness of the SDOF system, respectively. Eq. (2)

has another form which may be used in this work:

$$\omega_n^2 x + 2\omega_n \zeta \dot{x} + \ddot{x} = -\ddot{x}_g \tag{3}$$

where $\omega_n$ and $\zeta$ are the natural angular frequency and damping ratio, which are obtained from $\omega_n^2 = k/m$ and $c/m = 2\omega_n\zeta$, respectively.

### *2.1. Linear SDOF systems*

In this part, we discuss the basics of LIM for a linear SDOF dynamic system. First, we recall the impulse definition from particle dynamics. The load impulse is defined as the product of force and time as follows (Meriam, 2012):

$$dI = f(t)dt \tag{4}$$

The total impulse of the load applied to the mass $m$, from $t_i$ to $t_{i+1}$, can be calculated as follows:

$$I = \int_{t_i}^{t_{i+1}} f(t)dt \tag{5}$$

Now, multiplying Eq. (2) by $dt$, the load impulse terms of vibration equation is appeared:

$$kxdt + c\dot{x}dt + m\ddot{x}dt = -m\ddot{x}_g dt \tag{6}$$

Each term of Eq. (6) corresponds to the impulse of a load component in the vibrating system. Noting that $\dot{x}dt = dx$ and integrating Eq. (6) from $i$ to $i+1$, we can write:

$$\int_{t_i}^{t_{i+1}} kxdt + \int_{x_i}^{x_{i+1}} cdx + \int_{t_i}^{t_{i+1}} m\ddot{x}dt = \int_{t_i}^{t_{i+1}} -m\ddot{x}_g dt \tag{7}$$

Assuming a linear variation for $x$, and $\ddot{x}_g(t)$ with respect to time and employing trapezoidal integration rule (TIR) for the first, third, and the last terms of Eq. (7) gives:

$$\frac{kh}{2}(x_{i+1} + x_i) + c(x_{i+1} - x_i) + \frac{mh}{2}(\ddot{x}_{i+1} + \ddot{x}_i) = -\frac{mh}{2}(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}) \tag{8}$$

where $i$ and $i+1$ indices represent the value of each quantity at $t_i$ and $t_{i+1}$, respectively. The step size is also denoted by $h = t_{i+1} - t_i$. From now on, this notation will be employed throughout the context. The recursive relationship for computing $x_{i+1}$ can be obtained from Eq. (8), as follows:

$$x_{i+1} = x_i - \frac{mh}{2c}\left[\left(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}\right) + \omega_n^2\,(x_i + x_{i+1}) + (\ddot{x}_{i+1} + \ddot{x}_i)\right] \tag{9}$$

where $\omega_n^2 = k/m$. In a similar manner, using $\ddot{x}dt = d\dot{x}$ in the third term of Eq. (6), we can integrate it as:

$$\int_{t_i}^{t_{i+1}} kxdt + \int_{t_i}^{t_{i+1}} c\dot{x}dt + \int_{\dot{x}_i}^{\dot{x}_{i+1}} md\dot{x} = \int_{t_i}^{t_{i+1}} -m\ddot{x}_g dt \tag{10}$$

Use of TIR results in the following equation:

$$\frac{kh}{2}(x_{i+1} + x_i) + \frac{ch}{2}(\dot{x}_{i+1} + \dot{x}_i) + m(\dot{x}_{i+1} - \dot{x}_i) = -\frac{h}{2}(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}) \tag{11}$$

Solving Eq. (11) for $\dot{x}_{i+1}$, the recursive relation for the velocity is obtained:

$$\dot{x}_{i+1} = \dot{x}_i - \frac{h}{2}\left[\left(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}\right) + \omega_n^2\,(x_{i+1} + x_i) + 2\omega_n\zeta\,(\dot{x}_{i+1} + \dot{x}_i)\right] \tag{12}$$

where $2\omega_n\zeta$ is used for $c/m$. Eq. (9) and (12) construct the base relations for LIM computation. Prediction response at each instance could be provided by linear approximation of $x_{i+1}$ and $\dot{x}_{i+1}$ at $t_{i+1}$:

$$x_{i+1} = x_i + \dot{x}_i h \tag{13}$$

$$\dot{x}_{i+1} = \dot{x}_i + \ddot{x}_i h \tag{14}$$

Acceleration should be directly determined by the governing DEOM given by Eq. (3):

$$\ddot{x}_{i+1} = -(\ddot{x}_{g,i+1} + \omega_n^2\, x_{i+1} + 2\omega_n\zeta\dot{x}_{i+1}) \tag{15}$$

Stepwise algorithm of LIM for dynamic analysis of damped SDOF systems is summarized in Table 1.

### *2.2. Linear MDOF systems*

Here, the proposed LIM is generalized to linear MDOF structural systems. This is simply carried out by vectorizing the relationships extended for the linear SDOF systems. First, the governing DEOM for the MDOF systems subjected to $F(t)$ external dynamic load is expressed as:

$$[k]\{x\} + [c]\{\dot{x}\} + [m]\{\ddot{x}\} = \{F(t)\} \tag{16}$$

where $[m]$, $[c]$, and $[k]$ are the mass, damping, and stiffness matrices of the MDOF system, respectively. $\{F(t)\}$ is the external dynamic load vector. For MDOF structural system subjected to support excitation, $\{\ddot{x}_g\}$, DEOM can be written as follows:

$$[k]\{x\} + [c]\{\dot{x}\} + [m]\{\ddot{x}\} = -[m]\{\ddot{x}_g\} \tag{17}$$

Similar to SDOF systems, the load impulse for MDOF system is defined as the product of the force vector $\{F(t)\}$ and time differential of $dt$:

$$\{dI\} = \{f(t)\}\,dt \tag{18}$$

Then, the total impulse of the load applied to the MDOF structure can be given by:

$$\{dI\} = \int_{t_i}^{t_{i+1}} \{f(t)\}\,dt \tag{19}$$

Multiplying Eq. (17) by $dt$, we have:

$$[k]\{x\}dt + [c]\{\dot{x}\}dt + [m]\{\ddot{x}\}dt = -[m]\{\ddot{x}_g\}dt \tag{20}$$

**Table 1.** Stepwise algorithm of LIM for estimating the seismic response of linear damped SDOF systems

1. Choose the time increment $h$ and initialize with: *
$$i = 1,\ t_1 = 0,\ x_1 = x(0),\ \dot{x}_1 = \dot{x}(0),\ \ddot{x}_1 = -(\ddot{x}_{g,1} + 2\omega_n\zeta\dot{x}_1 + \omega_n^2 x_1)$$
2. Predict the response at time instance $i + 1$:
$$x_{i+1} \cong x_i + \dot{x}_i h$$
$$\dot{x}_{i+1} \cong \dot{x}_i + \ddot{x}_i h$$
3. Update the response at time instance $i + 1$: *
$$\ddot{x}_{i+1} = -(\ddot{x}_{g,i+1} + \omega_n^2 x_{i+1} + 2\omega_n\zeta\,\dot{x}_{i+1})$$
$$x_{i+1} = x_i - \frac{mh}{2c}\left(\left(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}\right) + \omega_n^2(x_{i+1} + x_i) + (\ddot{x}_{i+1} + \ddot{x}_i)\right)$$
$$\dot{x}_{i+1} = \dot{x}_i - \frac{h}{2}\left(\left(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}\right) + \omega_n^2(x_{i+1} + x_i) + 2\omega_n\zeta\,(\dot{x}_{i+1} + \dot{x}_i)\right)$$
4. Repeat steps 2 and 3 until none of the precision criteria is met. **
5. Set $i \to i + 1$ and repeat steps 2 to 5 for the next time instances.

*For the case of external loads, use the followings instead of support excitation in steps 1 and 3, respectively:
$$\ddot{x}_1 = m^{-1}\left(F_1 - (kx_1 + c\dot{x}_1)\right)$$
$$\ddot{x}_{i+1} = m^{-1}\left(F_{i+1} - (kx_{i+1} + c\dot{x}_{i+1})\right)$$
**Two or three iterations often yield satisfactorily accurate response.

***Important note***: Impulse load method does not yield reliable results for undamped systems and the systems for which $c \cong 0$.

Recalling $\{\dot{x}\}dt = d\{x\}$ and integrating Eq. (20) from $t_i$ to $t_{i+1}$, we get:

$$\int_{t_i}^{t_{i+1}} [k]\{x\}dt + \int_{\{x_i\}}^{\{x_{i+1}\}} [c]d\{x\} + \int_{t_i}^{t_{i+1}} [m]\{\ddot{x}\}dt = \int_{t_i}^{t_{i+1}} -[m]\{\ddot{x}_g\}dt \tag{21}$$

In linear MDOF systems, we assume that the mass, damping, and stiffness matrices all remain unchanged throughout vibration. Therefore, $[k]$, $[c]$, and $[m]$ can be treated as constant values in integration. Now, using TIR, we obtain:

$$\frac{h}{2}[k](\{x_{i+1}\} + \{x_i\}) + [c](\{x_{i+1}\} - \{x_i\}) + \frac{h}{2}[m](\{\ddot{x}_{i+1}\} + \{\ddot{x}_i\}) = -\frac{h}{2}[m]\left(\{\ddot{x}_{g,i+1}\} + \{\ddot{x}_{g,i}\}\right) \tag{22}$$

Solving $\{x_{i+1}\}$, the required relation for computing displacement in iteration process of LIM is obtained:

$$\{x_{i+1}\} = \{x_i\} - \frac{h}{2}[c]^{-1}[m]\Big(\left(\{\ddot{x}_{g,i+1}\} + \{\ddot{x}_{g,i}\}\right) + [m]^{-1}[k]\,(\{x_{i+1}\} + \{x_i\}) + (\{\ddot{x}_{i+1}\} + \{\ddot{x}_i\})\Big) \tag{23}$$

To extract the required recursive relation for $\{\dot{x}_{i+1}\}$, we first substitute $\{\ddot{x}\}dt$ for $\{d\dot{x}\}$ in Eq. (20) and integrate it from $i$ to $i + 1$:

$$\int_{t_i}^{t_{i+1}} [k]\{x\}dt + \int_{t_i}^{t_{i+1}} [c]\{\dot{x}\}dt + \int_{\{\dot{x}_i\}}^{\{\dot{x}_{i+1}\}} [m]\{d\dot{x}\} = \int_{t_i}^{t_{i+1}} -[m]\{\ddot{x}_g\}dt \tag{24}$$

Applying TIR to estimate the first, second, and last integrals, we obtain:

$$\frac{h}{2}[k](\{x_{i+1}\} + \{x_i\}) + \frac{h}{2}[c](\{\dot{x}_{i+1}\} + \{\dot{x}_i\}) + [m](\{\dot{x}_{i+1}\} + \{\dot{x}_i\}) = -\frac{h}{2}[m](\{\ddot{x}_{g,i+1}\} + \{\ddot{x}_{g,i}\}) \tag{25}$$

It gives the iteration formula required for $\{\dot{x}_{i+1}\}$:

$$\{\dot{x}_{i+1}\} = \{\dot{x}_i\} - \frac{h}{2}\Big((\{\ddot{x}_{g,i+1}\} + \{\ddot{x}_{g,i}\}) + [m]^{-1}[k](\{x_{i+1}\} + \{x_i\}) + [m]^{-1}[c](\{\dot{x}_{i+1}\} + \{\dot{x}_i\})\Big) \tag{26}$$

Eqs. (21) and (24) constitute the base relations for iteration process in LIM. Prediction is performed by linear estimation of $\{x_{i+1}\}$, and $\{\dot{x}_{i+1}\}$:

$$\{x_{i+1}\} = \{x_i\} + h\,\{\dot{x}_i\} \tag{27}$$

$$\{\dot{x}_{i+1}\} = \{\dot{x}_i\} + h\,\{\ddot{x}_i\} \tag{28}$$

Acceleration vector is directly calculated from the governing DEOM as follows:

$$\{\ddot{x}_{i+1}\} = -\big(\{\ddot{x}_{g,i+1}\} + [m]^{-1}[k]\{x_{i+1}\} + [m]^{-1}[c]\{\dot{x}_{i+1}\}\big) \tag{29}$$

It is noted that the proposed LIM needs the MDOF structure to have an invertible damping matrix.

**Table 2.** Stepwise algorithm of *LIM* for estimating the seismic response of linear damped *MDOF* systems

1. Choose the time increment $h$ and initialize with: *

   $i = 1,\ t_1 = 0,\ \{x_1\} = \{x(0)\},\ \{\dot{x}_1\} = \{\dot{x}(0)\},\ \{\ddot{x}_1\} = -\big(\{\ddot{x}_{g,1}\} + [m]^{-1}[k]\{x_1\} + [m]^{-1}[c]\{\dot{x}_1\}\big)$
2. Predict the response at time instance $i + 1$

   $\{x_{i+1}\} \cong \{x_i\} + h\,\{\dot{x}_i\}$

   $\{\dot{x}_{i+1}\} \cong \{\dot{x}_i\} + h\,\{\ddot{x}_i\}$
3. Update the response at time instance $i + 1$: *

   $\{\ddot{x}_{i+1}\} = -\big(\{\ddot{x}_{g,i+1}\} + [m]^{-1}[k]\{x_{i+1}\} + [m]^{-1}[c]\{\dot{x}_{i+1}\}\big)$

   $\{x_{i+1}\} = \{x_i\} - \frac{h}{2}[c]^{-1}[m]\Big((\{\ddot{x}_{g,i+1}\} + \{\ddot{x}_{g,i}\}) + [m]^{-1}[k]\,(\{x_{i+1}\} + \{x_i\}) + (\{\ddot{x}_{i+1}\} + \{\ddot{x}_i\})\Big)$

   $\{\dot{x}_{i+1}\} = \{\dot{x}_i\} - \frac{h}{2}\Big((\{\ddot{x}_{g,i+1}\} + \{\ddot{x}_{g,i}\}) + [m]^{-1}[k](\{x_{i+1}\} + \{x_i\}) + [m]^{-1}[c](\{\dot{x}_{i+1}\} + \{\dot{x}_i\})\Big)$
4. Repeat steps 2 and 3 until none of the precision criteria is met in its vector form. **
5. Set $i \to i + 1$ and repeat step 2 to 5 for the next time instances.

*For the case of external loads, use the followings instead of support excitation in steps 1 and 3, respectively:

$\ddot{x}_1 = [m]^{-1}(F_1 - ([c]\{\dot{x}_1\} + [k]\{x_1\}))$

$\ddot{x}_{i+1} = [m]^{-1}\big(F_{i+1} - ([c]\{\dot{x}_{i+1}\} + [k]\{x_{i+1}\})\big)$

**Two or three iterations often satisfactorily yield accurate response.

***Important note***: Impulse load method does not yield reliable results for undamped systems for which [c] ≡ 0 or $Det([c]) \approx 0$.

Thus, LIM cannot be used to estimate the dynamic response of the undamped structures for which the determinant of the damping matrix approaches zero. The detailed steps of LIM for estimating the dynamic response of linear damped MDOF systems are summarized in Table 2.

## *2.3. Nonlinear SDOF systems*

The governing DEOM for a nonlinear SDOF system subjected to the ground acceleration $\ddot{x}_g(t)$ can be formulated as (Chopra, 2012):

$$F_s(x,\dot{x}) + F_d(x\,,\dot{x}) + m\ddot{x} = -m\ddot{x}_g \tag{30}$$

where $F_s(x,\dot{x})$ and $F_d(x,\dot{x})$ are the inelastic resisting and non-viscus damping forces, respectively. The values of $F_s$ and $F_d$ typically depend on the displacement and velocity time-histories of the nonlinear SDOF system in the previous instances. Multiplying Eq. (30) by $dt$, we have:

$$F_s(x,\dot{x})dt + F_d(x\,,\dot{x})dt + m\ddot{x}dt = -m\ddot{x}_g dt \tag{31}$$

Recalling $d\dot{x} = \ddot{x}dt$, we integrate Eq. (31) from $i$ to $i+1$:

$$\int_{t_i}^{t_{i+1}} F_s(x,\dot{x})dt + \int_{t_i}^{t_{i+1}} F_d(x,\dot{x})dt + \int_{\dot{x}_i}^{\dot{x}_{i+1}} m d\dot{x} = -\int_{t_i}^{t_{i+1}} m\ddot{x}_g dt \tag{32}$$

Approximating the first, second, and last integrals by TIR, we obtain:

$$\frac{h}{2}\left(F_{s,i+1} + F_{s,i}\right) + \frac{h}{2}\left(F_{d,i+1} + F_{d,i}\right) + m\left(\dot{x}_{i+1} - \dot{x}_i\right) = -\frac{mh}{2}\left(\ddot{x}_{g,i+1} + \ddot{x}_{g,i}\right) \tag{33}$$

Each term of Eq. (33) represents the total impulse loads related to the spring, friction, inertia, and the external force, respectively. Solving Eq. (33) for $\dot{x}_{i+1}$, we get:

$$\dot{x}_{i+1} = \dot{x}_i - \frac{h}{2m}\left(m\ddot{x}_{g,i+1} + F_{d,i+1} + F_{s,i+1}\right) - \frac{h}{2m}\left(m\ddot{x}_{g,i} + F_{d,i} + F_{s,i}\right) \tag{34}$$

Using Eq. (2), we can simplify Eq. (34) as:

$$\dot{x}_{i+1} = \dot{x}_i + \frac{h}{2}\left(\ddot{x}_i + \ddot{x}_{i+1}\right) \tag{35}$$

This formula is not a new one; however, it says that we can use available integration tools in computation of derivatives. Eq. (35) is known as Adams-Moulton method (AMM) in numerical calculus of solving ODEs. AMM benefits trapezoidal integration rule (TIR) in its body. In vibration analysis, the governing equation is a 2nd-order ODE. So, 2nd-order derivatives are available while computing the response. Based on this fact, the displacement is suggested to simply and accurately be estimated by corrected trapezoidal integration rule (CTIR), as follows:

$$x_{i+1} = x_i + \frac{h}{2}\left(\dot{x}_i + \dot{x}_{i+1}\right) + \frac{h}{12}\left(\ddot{x}_i - \ddot{x}_{i+1}\right) \tag{36}$$

Eqs. (33) and (34) should be coupled with the following equation obtained from the governing DEOM:

$$\ddot{x}_{i+1} = -m^{-1}\left(m\ddot{x}_{g,i+1} + F_{d,i+1} + F_{s,i+1}\right) \tag{37}$$

**Table 3.** Stepwise algorithm of LIM for estimating the seismic response of nonlinear SDOF systems

1. Choose the time increment $\Delta t$ and initialize with: *

$$t_1 = 0\ ,\ x_1 = x(0)\ ,\ \dot{x}_1 = \dot{x}(0)$$
$$F_{s,1} = F_s(x_1,\dot{x}_1)\ ,\ F_{d,1} = F_d(x_1,\dot{x}_1)$$
$$\ddot{x}_1 = -m^{-1}\left(ma_{g,1} + F_{s,1} + F_{d,1}\right)$$

2. Predict the response at time instance $i+1$:

$$x_{i+1} \cong x_i + \dot{x}_i h$$
$$\dot{x}_{i+1} \cong \dot{x}_i + \ddot{x}_i h$$

3. Update the response at time instance $i+1$: *

$$F_{s,i+1} = F_s(x_{i+1},\dot{x}_{i+1})\ ,\ F_{d,i+1} = F_d(x_{i+1},\dot{x}_{i+1})$$
$$\ddot{x}_{i+1} = -m^{-1}\left(m\ddot{x}_{g,i+1} + F_{s,i+1} + F_{d,i+1}\right)$$
$$\dot{x}_{i+1} = \dot{x}_i + \frac{h}{2}\left(\ddot{x}_i + \ddot{x}_{i+1}\right)$$
$$x_{i+1} = x_i + \frac{h}{2}\left(\dot{x}_i + \dot{x}_{i+1}\right) + \frac{h}{12}\left(\ddot{x}_i - \ddot{x}_{i+1}\right)$$

4. Repeat step 3 until none of the precision criteria is met. **
5. Set $i = i+1$ and repeat steps 2 to 5 for the next time instances.

*For the case of external loads instead of support excitation, use the followings in steps 1 and 3, respectively:

$$\ddot{x}_1 = m^{-1}\left(F_1 - \left(F_{s,1} + F_{d,1}\right)\right)$$
$$\ddot{x}_{i+1} = m^{-1}\left(F_{i+1} - \left(F_{s,i+1} + F_{d,i+1}\right)\right)$$

**Two or three iterations often satisfactorily yield accurate response.

The procedure of nonlinear LIM algorithm for estimating SDOF systems are summarized in Table 3. Notably, both damped and undamped systems are covered in this formulation. Nonlinear LIM is one of the simplest algorithms for dynamic analysis of structural systems. It is completely general to deal with various types of nonlinearities and external loadings.

### 2.4. Nonlinear MDOF systems

A great advantage of nonlinear LIM algorithm, presented for SDOF systems, is its generalizability to MDOF systems without any change in its formulation. In this regard, we merely present the vector form of the relations which is developed for SDOF systems. For the sake of convenience, the implementation steps of nonlinear LIM algorithm are provided in Table 4. Notably, there is no restriction on the properties of structural system and loading function while working with this algorithm.

### 2.5. Precision criteria

Precision check of solution is of high importance with numerical approaches. There are various tools for evaluating the accuracy and efficiency of the numerical methods. Some of the well-defined convergence criteria are as follows:
1. The residual of the governing DEOM can be basically identified as the unbalanced force. If the estimation precision is accurately enough, the absolute value of the governing DEOM approaches zero. Hence, the absolute value of the governing DEOM approaches zero. Hence, the absolute value of the residual function can suitably be employed to check the accuracy of the proposed method. The error in the form of the unbalanced forces for a SDOF system at a given time instance is as follows:

$$R_i = UF_i = \left|m\ddot{x}_i + c\dot{x}_i + kx_i + m\ddot{x}_{g,i}\right| \tag{38}$$

The residual function for a nonlinear SDOF system can be obtained in a similar manner as follows:

$$R_i = UF_i = \left|F_{s,i} + F_{d,i} + kx_i + m\ddot{x}_{g,i}\right| \tag{39}$$

2. For an arbitrary time-varying dynamic load function of $F(t)$, $m\ddot{x}_{g,i}$ must be replaced by $-F_i$ in Eqs. (38) and (39). The residual function at $j$th iteration of the $i$th time instance, $R_i^{(j)}$, must satisfy the following inequality:

$$\left|R_i^{(j)}\right| \le \varepsilon_r \tag{40}$$

where $\varepsilon_R$ is the tolerance parameter and selected a value ranging from $10^{-8}$ to $10^{-3}$.
3. Alternative criterion of accuracy level in numerical analysis is the displacement change of $\Delta x_i^{(j)} = x_i^{(j)} - x_i^{(j-1)}$. This criterion will be satisfied at $j$th iteration of $i$th time instance if:

$$\left|\Delta x_i^{(j)}\right| \le \varepsilon_d \tag{41}$$

where $\varepsilon_d$ is the displacement tolerance parameter and often selected a value ranging from $10^{-8}$ to $10^{-3}$.
4. The virtual work obtained by the residual force $R_i^{(j)} = R_i^{(j)} - R_i^{(j-1)}$ at the displacement change $\Delta x_i^{(j)} = x_i^{(j)} - x_i^{(j-1)}$ could be another choice. This criterion is satisfied if the value of virtual work stands less than $\varepsilon_w$:

$$\frac{1}{2}\left|\Delta x_i^{(j)} R_i^{(j)}\right| \le \varepsilon_w \tag{42}$$

**Table 4.** Stepwise algorithm of LIM for estimating the seismic response of nonlinear MDOF systems

1. Choose the time increment $h$ and initialize with: *

$$t_1 = 0 \ , \ \{x_1\} = \{x(0)\} \ , \ \{\dot{x}_1\} = \{\dot{x}(0)\}$$
$$\{F_{s,1}\} = \{F_s(x_1, \dot{x}_1)\} \ , \ \{F_{d,1}\} = \{F_d(x_1, \dot{x}_1)\}$$
$$\{\ddot{x}_1\} = -[m]^{-1}\left([m]\{a_{g,1}\} + \{F_{s,1}\} + \{F_{d,1}\}\right)$$

2. Predict the response at time instance $i + 1$:

$$\{x_{i+1}\} = \{x_i\} + h\,\{\dot{x}_i\}$$
$$\{\dot{x}_{i+1}\} = \{\dot{x}_i\} + h\,\{\ddot{x}_i\}$$

3. Update the response at time instance $i + 1$: **

$$\{F_{s,i+1}\} = \{F_s(x_{i+1}, \dot{x}_{i+1})\} \ , \ \{F_{d,i+1}\} = \{F_d(x_{i+1}, \dot{x}_{i+1})\}$$
$$\{\ddot{x}_{i+1}\} = -[m]^{-1}\left([m]\{\ddot{x}_{g,i+1}\} + \{F_{s,i+1}\} + \{F_{d,i+1}\}\right)$$
$$\{\dot{x}_{i+1}\} = \{\dot{x}_i\} + \frac{h}{2}\left(\{\ddot{x}_i\} + \{\ddot{x}_{i+1}\}\right)$$
$$\{x_{i+1}\} = \{x_i\} + \frac{h}{2}\left(\{\dot{x}_i\} + \{\dot{x}_{i+1}\}\right) + \frac{h}{12}\left(\{\ddot{x}_i\} - \{\ddot{x}_{i+1}\}\right)$$

4. Repeat step 3 until none of the precision criteria is met in its vector form. **
5. Set $i = i + 1$ and repeat steps 2 to 5 for the next time instances.

*For the case of external loads instead of support excitation, use the followings in steps 1 and 3, respectively:

$$\{\ddot{x}_1\} = [m]^{-1}\left(\{F_1\} - \left(\{F_{d,1}\} + \{F_{s,1}\}\right)\right)$$
$$\{\ddot{x}_{i+1}\} = [m]^{-1}\left(\{F_{i+1}\} - \left(\{F_{d,i+1}\} + \{F_{s,i+1}\}\right)\right)$$

**Two or three iterations often yield satisfactorily accurate response.

The value for the tolerance $\varepsilon_w$ is often selected near the computer precision, i.e., the smallest positive value recognizable by the computer because the left-hand side of Eq. (42) is a product of two infinitesimal quantities (Chopra, 2012). In the next part, the accuracy of the proposed LIM is mostly evaluated for several example case studies using the first and second criteria.

## 3. Numerical examples

In this part, the accuracy and efficiency of the proposed procedures in estimating the seismic response of different linear and nonlinear systems (hereafter referred to M1 to M5 models) under dynamic loads is investigated. The values of mass ($m$), natural fundamental period ($T$), damping ratio ($\zeta$), and stiffness (k) related to each model are presented in Table 5. Displacement time-history response for each model is first calculated through the proposed LIM and then computed by the robust Duhamel integral, Newmark-β, and Wilson-θ methods. In the analyses, the time increment is assumed equal to the ground motion sampling time-step, first. Then, in order to show the convergence of the new method, the analysis step is decreased to ten times smaller than the ground motion recording step. Time-acceleration of the two selected earthquake records, together with their recording time step ($h_r$), and peak ground accelerations (PGAs) have been indicated in Fig. 2.

### 3.1. Numerical analysis of linear SDOF system

Figs. 3 and 4 compare the displacement time-histories of models M1 and M2. They are estimated by the proposed LIM algorithm and those given by the Duhamel integral and Newmark-β methods under El Centro and Kobe earthquakes, respectively. As shown in Figs. 3 and 4, except the time instances ranging from 3 to 11 seconds in the M1 model, LIM estimates the displacement response of the M1 and M2 models with high accuracy. This deficiency is originated from using larger step in analysis, which is not appropriate for the procedure. As evident in Figs. 3 and 4, the displacement demands obtained from the proposed method fully matches those from the Duhamel integral and Newmark-β approaches. It is noted that even when time increment is relatively large, the peak values of the responses obtained by the proposed method well matches those from the conventional methods. The discrepancies between the results are fully eliminated when a sufficiently small step (i.e., 10% of the ground motion time-step) is used. Figs. 5 and 6 compare the response time-histories of the M1 and M2 models which are estimated by finer mesh.

As shown in these figures, the proposed method can appropriately estimate the dynamic response of the M1 and M2 models with sufficient accuracy in all cases. Tables 6 and 7 also compare the peak responses obtained from various methods. The results clearly show that there is good agreement between the results obtained from the proposed and Newmark-β methods. A quantitative comparison of the results obtained from the proposed LIM and the other methods has been provided in Tables 8 and 9 for the M1 and M2 models, respectively. The estimation errors from the proposed procedure have also been presented in these tables. Deviation from the conventional methods is almost less than 1%.

**Table 5.** Dynamic properties of the structural models in the examples.

| Model | Type | Behavior | Excitation | $m$ (kip.sec²/in) | $c$ | $T$ (sec) | $k$ (kips/in) |
|---|---|---|---|---|---|---|---|
| M1 | SDOF | Linear | El-Cento Record | 1 | 0.02 | 0.5 | 157.9 |
| M2 | SDOF | Linear | Kobe Record | 1 | 0.05 | 0.3 | 438.7 |
| M3 | MDOF | Linear | External Force | $\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$ | $\begin{bmatrix}0.36 & -0.18\\-0.18 & 0.18\end{bmatrix}$ | $\begin{bmatrix}3.475\\2.421\end{bmatrix}$ | $\begin{bmatrix}6 & -2\\-2 & 8\end{bmatrix}$ |
| M4 | SDOF | Nonlinear | El-Cento Record | 1 | 0.02 | 0.5 | 157.9 |
| M5 | SDOF | Nonlinear | Kobe Record | 1 | 0.05 | 0.3 | 438.7 |

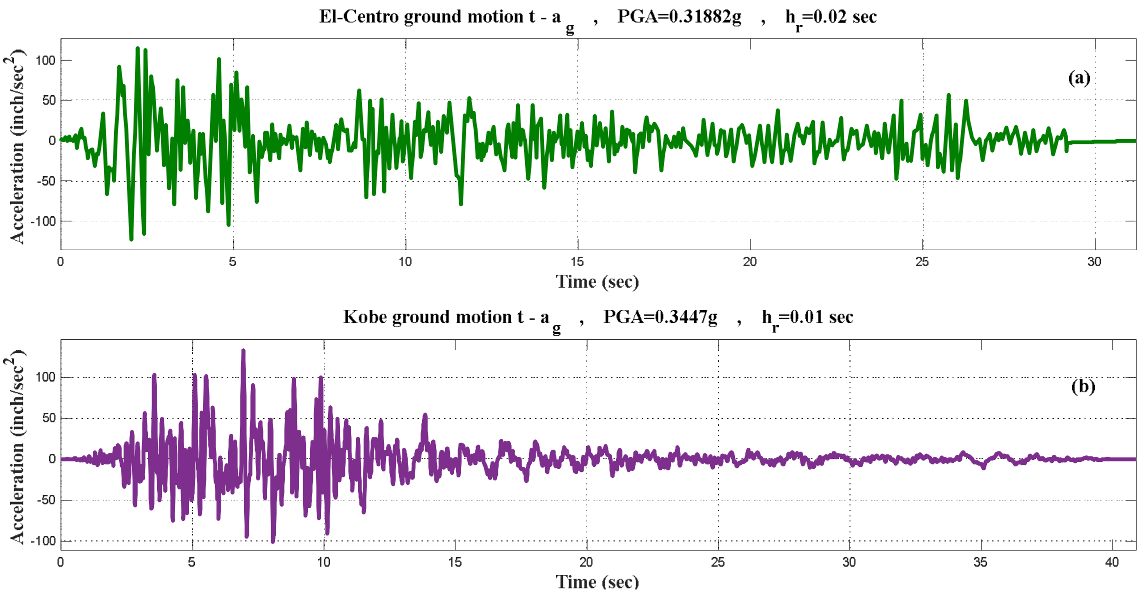


**Fig. 2.** The earthquake ground motion records used for dynamic analysis of M1, M2, M4, and M5 models.

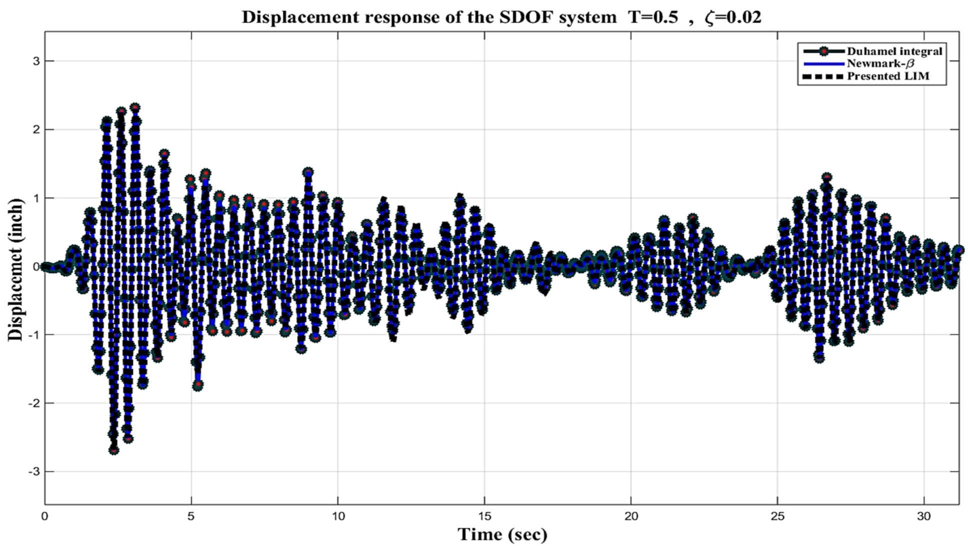


**Fig. 3**. Displacement time-histories obtained from LIM, Duhamel, and Newmark-β methods for M1 model under El Centro ground motion record. The analysis time step is $h=$ 0.02 sec.

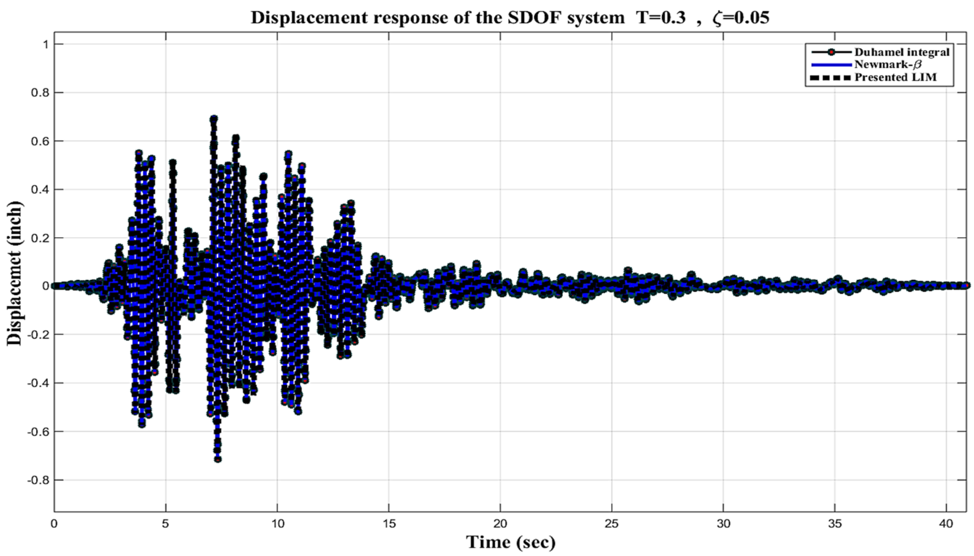


**Fig. 4**. Displacement time-histories obtained from LIM, Duhamel, and Newmark-β methods for M2 model under Kobe ground motion record. The analysis time step is $h$=0.01 sec.

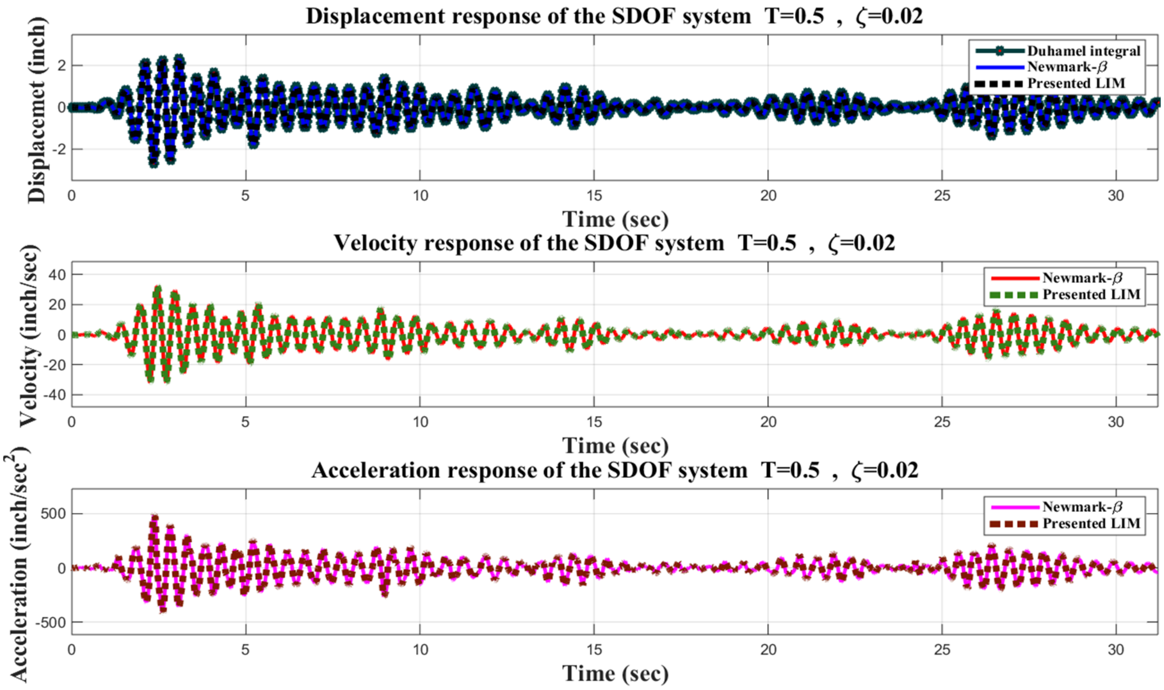


**Fig. 5**. Response time-histories for M1 model under El Centro ground motion record. The analysis time step is $h = 0.002\ sec$.

According to the results, the main advantages of the proposed procedure in estimating the seismic response of the linear damped SDOF systems are as follows: (1) the estimation errors from LIM gradually decays in the proceeding instances; (2) Unlike similar approaches, fewer initialization parameters are required by LIM algorithm; (3) A fast convergence rate is provided by LIM and a satisfactory level of precision is offered; (4) Because LIM computation is of first order [O], it works faster than the Duhamel integral which uses the computation of order two $[O]^2$.

### 3.2. Numerical analysis of linear MDOF system

In this example, the proposed LIM is checked on a linear MDOF structural model, which is depicted in Fig. 7. For this purpose, the dynamic response of the M3 model is determined by LIM and then it is computed and compared with the results from the most known Wilson-θ method devised for analyzing MDOF systems.

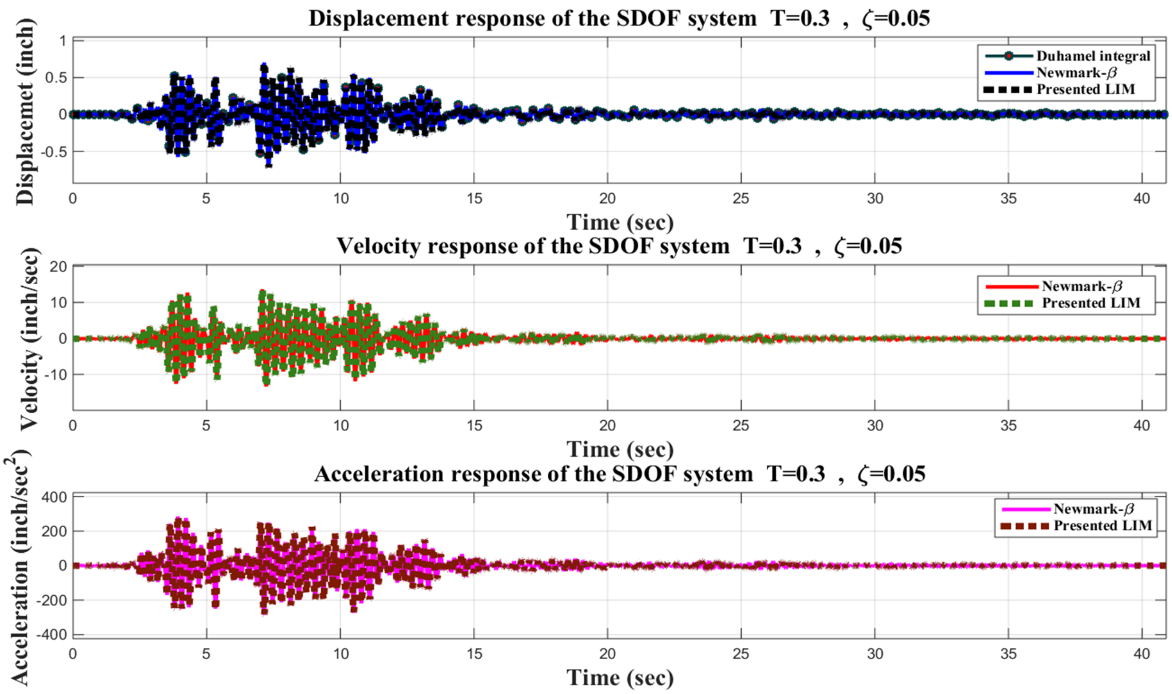


**Fig. 6**. Response time-histories for M2 model under Kobe ground motion record. The analysis time step is $h = 0.001$ sec.

**Table 6.** Summary results obtained from LIM, Duhamel, and Newmark-β methods for M1 model.

| Method | $\ddot{x}_{max}$ (in/sec2) | $\dot{x}_{max}$ (in/sec) | $x_{max}$ (in) | Analysis time (sec) | # of iterations | Time step $h$ (sec) |
|---|---|---|---|---|---|---|
| Duhamel | - | - | 2.6881 | 13.611 | 1 | 0.002 |
| Newmark-β | 486.42 | 32.265 | 2.2880 | 0.002 | 1 | 0.002 |
| LIM | 487.11 | 32.203 | 2.6878 | 0.004 | 2 | 0.002 |

**Table 7.** Summary results obtained from LIM, Duhamel, and Newmark-β methods for M2 model.

| Method | $\ddot{x}_{max}$ (in/sec$^2$) | $\dot{x}_{max}$ (in/sec) | $x_{max}$ (in) | Analysis time (sec) | # of iterations | Time step $h$ (sec) |
|---|---|---|---|---|---|---|
| Duhamel | - | - | 0.7127 | 96.66 | 1 | 0.001 |
| Newmark-β | 284.90 | 13.637 | 0.7127 | 0.005 | 1 | 0.001 |
| LIM | 284.93 | 13.654 | 0.7124 | 0.011 | 2 | 0.001 |

**Table 8**. Results obtained from LIM, Duhamel, and Newmark-β methods for M1 model under El Centro ground motion record. The analysis time step is $h = 0.002\ sec$.

| $t_i$ (sec) | Displacement $x_i$ $(in)$ | | | Velocity $\dot{x}_i (in/sec)$ | | Acceleration $\ddot{x}_i$ $(in/sec^2)$ | | Error | |
|---|---|---|---|---|---|---|---|---|---|
| | Duhamel | Newmark-β | LIM | Newmark-β | LIM | Newmark-β | LIM | $x_i$ respect to Duhamel (%) | UF of LIM |
| 0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.00 | <1e-5 |
| 1.2 | -0.085 | -0.085 | -0.085 | -3.358 | -3.358 | -11.875 | -11.877 | 0.00 | <1e-5 |
| 2.4 | -2.147 | -2.147 | -2.147 | 22.527 | 22.522 | 443.443 | 443.474 | 0.01 | <1e-5 |
| 3.6 | 1.394 | 1.394 | 1.394 | -3.407 | -3.393 | -215.471 | -215.495 | 0.01 | <1e-5 |
| 4.8 | -0.618 | -0.618 | -0.617 | 7.892 | 7.877 | 142.881 | 142.818 | 0.13 | <1e-5 |
| 6.0 | 0.814 | 0.814 | 0.813 | -8.541 | -8.529 | -125.880 | -125.793 | 0.14 | <1e-5 |
| 7.2 | -0.977 | -0.976 | -0.975 | -2.589 | -2.597 | 165.131 | 165.011 | 0.16 | <1e-5 |
| 8.4 | 0.558 | 0.557 | 0.556 | 9.138 | 9.141 | -80.702 | -80.552 | 0.34 | <1e-5 |
| 9.6 | 0.294 | 0.295 | 0.296 | -11.785 | -11.781 | -59.903 | -60.067 | 0.71 | <1e-5 |
| 10.8 | -0.456 | -0.457 | -0.457 | 5.197 | 5.187 | 63.523 | 63.635 | 0.31 | <1e-5 |
| 12.0 | -0.229 | -0.229 | -0.229 | 8.997 | 9.009 | 7.770 | 7.768 | 0.00 | <1e-5 |
| 13.2 | 0.176 | 0.177 | 0.177 | 0.495 | 0.490 | -23.556 | -23.629 | 0.51 | <1e-5 |
| 14.4 | -0.828 | -0.828 | -0.828 | -3.561 | -3.563 | 126.406 | 126.462 | 0.08 | <1e-5 |
| 15.6 | 0.202 | 0.202 | 0.202 | 1.543 | 1.548 | -43.505 | -43.507 | 0.00 | <1e-5 |
| 16.8 | 0.217 | 0.217 | 0.217 | -1.956 | -1.959 | -41.909 | -41.940 | 0.18 | <1e-5 |
| 18.0 | -0.007 | -0.007 | -0.007 | 1.246 | 1.246 | 1.036 | 1.077 | NA** | <1e-5 |
| 19.2 | -0.129 | -0.128 | -0.128 | -2.057 | -2.057 | 10.115 | 10.081 | 0.39 | <1e-5 |
| 20.4 | -0.438 | -0.438 | -0.438 | 2.088 | 2.086 | 94.361 | 94.382 | 0.07 | <1e-5 |
| 21.6 | 0.432 | 0.432 | 0.432 | 5.118 | 5.122 | -78.655 | -78.641 | 0.05 | <1e-5 |
| 22.8 | -0.329 | -0.329 | -0.328 | -2.408 | -2.412 | 65.069 | 65.009 | 0.24 | <1e-5 |
| 24.0 | -0.018 | -0.018 | -0.019 | 0.446 | 0.447 | 17.305 | 17.367 | 4.4 | <1e-5 |
| 25.2 | 0.547 | 0.547 | 0.547 | 3.968 | 3.971 | -76.050 | -76.081 | 0.07 | <1e-5 |
| 26.4 | -1.282 | -1.282 | -1.282 | -5.687 | -5.692 | 197.229 | 197.196 | 0.03 | <1e-5 |
| 27.6 | 0.406 | 0.406 | 0.405 | 11.931 | 11.936 | -75.580 | -75.477 | 0.32 | <1e-5 |
| 28.8 | 0.001 | 0.002 | 0.002 | -7.875 | -7.875 | 17.593 | 17.457 | NA** | <1e-5 |
| 30.0 | -0.177 | -0.178 | -0.179 | 3.643 | 3.637 | 27.791 | 27.890 | 0.67 | <1e-5 |
| 31.2 | 0.237 | 0.237 | 0.237 | -1.1888 | -1.180 | -36.854 | -36.887 | 0.17 | <1e-5 |

*$FU_i$ is the unbalanced force of the proposed analysis method.
**NA: Not Applicable since the value of $x_i$ is under computation precision level which is approximately 1e-3.

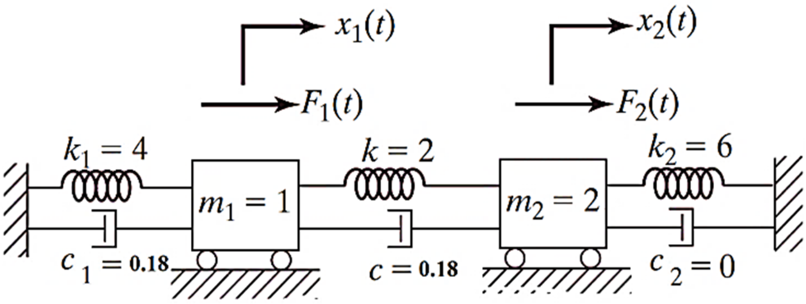


**Fig. 7**. M3 model with two degrees of freedom subjected to the external load $\{F(t)\}$ (Rao, 2017).

**Table 9.** Results obtained from LIM, Duhamel, and Newmark-β methods for M2 model under Kobe ground motion record. The analysis time step is $h = 0.001\ sec$

| $t_i$ (sec) | Displacement $x_i\ (in)$ | | | Velocity $\dot{x}_i (in/sec)$ | | Acceleration $\ddot{x}_i\ (in/sec^2)$ | | Error | |
|---|---|---|---|---|---|---|---|---|---|
| | Duhamel | Newmark-β | LIM | Newmark-β | LIM | Newmark-β | LIM | $x_i$ respect to Duhamel (%) | UF of LIM |
| 0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0 | <1e-5 |
| 1.2 | -0.004 | -0.004 | -0.004 | 0.063 | 0.063 | 2.308 | 2.3080 | 0 | <1e-5 |
| 2.4 | 0.083 | 0.083 | 0.083 | 0.509 | 0.509 | -3.835 | -3.8360 | 0 | <1e-5 |
| 3.6 | -0.431 | -0.431 | -0.431 | -6.311 | -6.312 | 171.713 | 171.696 | 0 | <1e-5 |
| 4.8 | -0.132 | -0.132 | -0.132 | -1.53 | -1.532 | 115.862 | 115.844 | 0 | <1e-5 |
| 6.0 | 0.225 | 0.225 | 0.225 | 0.692 | 0.691 | -37.2060 | -37.227 | 0 | <1e-5 |
| 7.2 | 0.395 | 0.396 | 0.396 | -11.242 | -11.241 | -134.844 | -134.869 | 0.25 | <1e-5 |
| 8.4 | 0.292 | 0.292 | 0.292 | 7.448 | 7.450 | -95.529 | -95.531 | 0 | <1e-5 |
| 9.6 | -0.014 | -0.014 | -0.014 | 3.901 | 3.902 | 9.901 | 9.9360 | 0 | <1e-5 |
| 10.8 | 0.422 | 0.422 | 0.422 | -3.233 | -3.230 | -207.728 | -207.718 | 0 | <1e-5 |
| 12.0 | -0.168 | -0.168 | -0.168 | 1.575 | 1.576 | 54.839 | 54.837 | 0 | <1e-5 |
| 13.2 | -0.119 | -0.119 | -0.12 | 6.111 | 6.111 | 66.905 | 66.93 | 0.84 | <1e-5 |
| 14.4 | 0.126 | 0.126 | 0.126 | 0.052 | 0.052 | -36.087 | -36.069 | 0 | <1e-5 |
| 15.6 | -0.005 | -0.005 | -0.005 | -0.519 | -0.519 | 3.700 | 3.6990 | 0 | <1e-5 |
| 16.8 | -0.090 | -0.090 | -0.090 | 0.499 | 0.499 | 33.573 | 33.572 | 0 | <1e-5 |
| 18.0 | -0.039 | -0.039 | -0.039 | 1.304 | 1.304 | -0.406 | -0.4000 | 0 | <1e-5 |
| 19.2 | -0.002 | -0.002 | -0.002 | 0.939 | 0.939 | -8.635 | -8.628 | 0 | <1e-5 |
| 20.4 | -0.005 | -0.005 | -0.005 | -0.266 | -0.266 | 6.769 | 6.773 | 0 | <1e-5 |
| 21.6 | -0.005 | -0.005 | -0.005 | 0.821 | 0.821 | 13.844 | 13.845 | 0 | <1e-5 |
| 22.8 | 0.03 | 0.03 | 0.03 | -0.228 | -0.228 | -1.478 | -1.479 | 0 | <1e-5 |
| 24.0 | 0.027 | 0.027 | 0.027 | -0.726 | -0.726 | -10.69 | -10.693 | 0 | <1e-5 |
| 25.2 | 0.002 | 0.002 | 0.002 | -0.622 | -0.622 | -1.585 | -1.588 | 0 | <1e-5 |
| 26.4 | 0.019 | 0.019 | 0.019 | -0.718 | -0.718 | -11.682 | -11.687 | 0 | <1e-5 |
| 27.6 | 0.021 | 0.021 | 0.021 | -0.557 | -0.557 | -7.882 | -7.886 | 0 | <1e-5 |
| 28.8 | 0.007 | 0.007 | 0.007 | 0.415 | 0.415 | 0.221 | 0.222 | 0 | <1e-5 |
| 30.0 | -0.003 | -0.003 | -0.003 | -0.055 | -0.055 | 2.077 | 2.077 | 0 | <1e-5 |
| 31.2 | -0.011 | -0.011 | -0.011 | -0.064 | -0.064 | 2.224 | 2.223 | 0 | <1e-5 |
| 32.4 | 0.022 | 0.022 | 0.022 | -0.164 | -0.164 | -10.475 | -10.476 | 0 | <1e-5 |
| 33.6 | -0.012 | -0.012 | -0.012 | -0.105 | -0.105 | 5.477 | 5.475 | 0 | <1e-5 |
| 34.8 | 0.017 | 0.017 | 0.017 | 0.141 | 0.141 | -4.666 | -4.666 | 0 | <1e-5 |
| 36.0 | -0.014 | -0.014 | -0.014 | 0.078 | 0.078 | 3.034 | 3.034 | 0 | <1e-5 |
| 37.2 | -0.024 | -0.024 | -0.024 | 0.227 | 0.227 | 9.390 | 9.391 | 0 | <1e-5 |
| 38.4 | -0.007 | -0.007 | -0.007 | 0.114 | 0.114 | 3.637 | 3.638 | 0 | <1e-5 |
| 39.6 | -0.001 | -0.001 | -0.001 | 0.036 | 0.036 | 0.690 | 0.690 | 0 | <1e-5 |
| 40.8 | -0.001 | -0.001 | -0.001 | 0.017 | 0.017 | 0.217 | 0.217 | 0 | <1e-5 |

*$FU_i$ is the unbalanced force value of the proposed analysis method.

**Table 10.** Summary results obtained from LIM and Wilson-θ methods for M3 model

| Method | $\ddot{x}_{max}$ (in/sec²) | $\dot{x}_{max}$ (in/sec) | $x_{max}$ (in) | Analysis time (sec) | # of iterations | Time step $h$ (sec) |
|---|---|---|---|---|---|---|
| Wilson-θ | $\begin{bmatrix}4.014\\5.000\end{bmatrix}$ | $\begin{bmatrix}1.918\\2.455\end{bmatrix}$ | $\begin{bmatrix}1.3143\\2.5848\end{bmatrix}$ | 0.003 | 1 | 0.0242 |
| LIM | $\begin{bmatrix}4.017\\5.000\end{bmatrix}$ | $\begin{bmatrix}1.913\\2.482\end{bmatrix}$ | $\begin{bmatrix}1.3176\\2.5848\end{bmatrix}$ | 0.005 | 2 | 0.0242 |

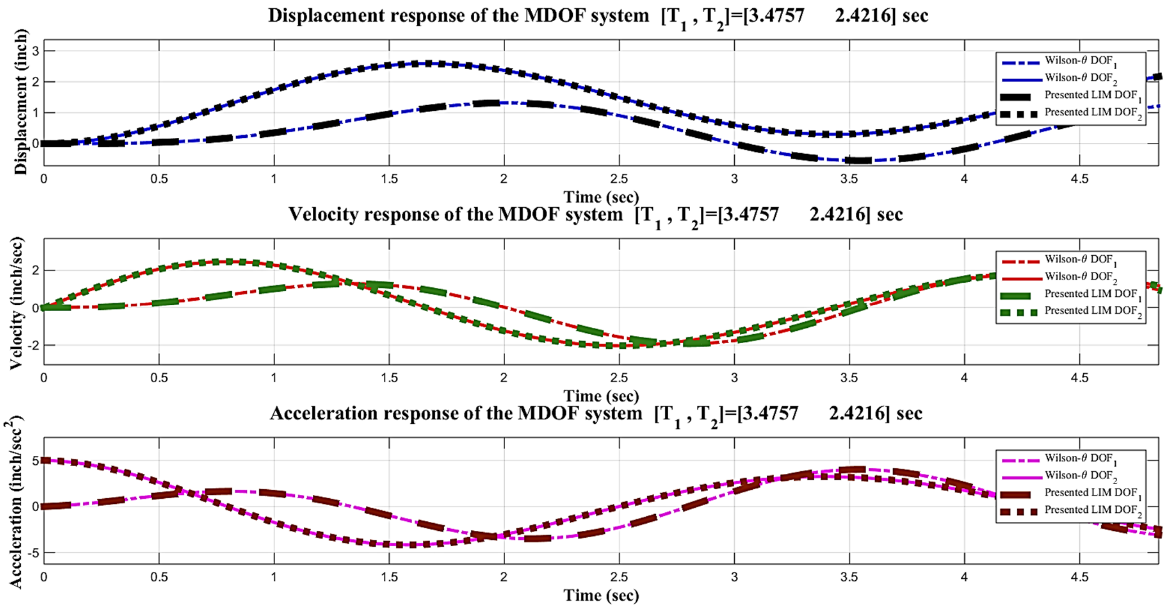


**Fig. 8.** Response time-histories for M3 model. The analysis time step is $h$ = 0.02416 sec.

Using the structural dynamics fundamentals, the mass (m), damping (c), and stiffness (k) matrices of the M3 model are computed as follows:

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} kip.sec^2/in \quad (43)$$

$$[c] = \begin{bmatrix} c_1 + c & -c \\ -c & c_2 + c \end{bmatrix} = \begin{bmatrix} 0.36 & -0.18 \\ -0.18 & 0.18 \end{bmatrix} \quad (44)$$

$$[k] = \begin{bmatrix} k_1 + k & -k \\ -k & k_2 + k \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix} kip/in \quad (45)$$

The system starts vibration from zero initial conditions. Forced vibration is then generated by the external load of:

$$\{F(t)\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} kip \quad (46)$$

According to the classic relations in structural dynamics, the natural periods for the first and second modes of vibration for the M3 model are determined as $T_1$ = 3.475 $sec$ and $T_2$ = 2.4216 $sec$, respectively. One percent of the minimum periods (i.e., $h = 0.01T_{min}$ = 0.02416 $sec$) is then assumed as the time increment for the current dynamic analyses. Fig. 8 compares time histories estimated by LIM with those given by the Wilson-θ method. High coincidence is evident between the results and those given by the Wilson-θ method. Small errors with the proposed method clearly indicate that LIM can satisfactorily estimate the response of elastic MDOF systems subjected to external dynamic loads.

*Table 10 summarizes the results to transmit a sense of efficiency and precision level of two methods.*LIM is not as fast as Wilson-θ method; but it works precisely enough to deal with linear MDOF systems. Detailed comparison is possible through the time histories reported in Tables 11 and 12. According to Tables 11 and 12, the response time-histories from LIM and Wilson-θ methods are almost the same.

### *3.3. Numerical analysis of nonlinear SDOF system*

In this example, the performance of the proposed procedure is evaluated for the nonlinear M4 and M5 SDOF models. The response of the models is determined, whereas a nonlinear behavior of the force-deformation is included. An elastic-perfectly-plastic behavior is assumed for the resisting force component (see Fig. 9). Maximum value of the forces (denoted by $f_o = kx_{e,max}$ in technical context (Chopra, 2012)) for M4 and M5 models are computed as 424.47 kips and 314.27 kips, respectively. It is noted that $x_{e,max}$ is the maximum displacement of the system which has linear resisting force in its spring component. Its value is extracted from the analysis results of M1 and M2 models. The yielding forces of the M4 and M5 models are assumed half of the corresponding maximum elastic force, i.e., 212.24 kips and 157.13 kips for M4 and M5 models, respectively.

Due to the inelastic behavior of the system, the seismic response of M4 and M5 models cannot be estimated by Duhamel integration because this procedure is only limited to the linear systems for which the superposition rule is valid. Thus, the robust nonlinear Newmark-β approach is used in this study to assess the performance of LIM for the nonlinear SDOF systems. Figs. 10 and 11 compare the response time-histories obtained from LIM with those given by the nonlinear Newmark-β method for the M4 and M5 models, respectively. Tables 13 and 14 also compare the peaks obtained from LIM with those from Duhamel and Newmark-β approaches.

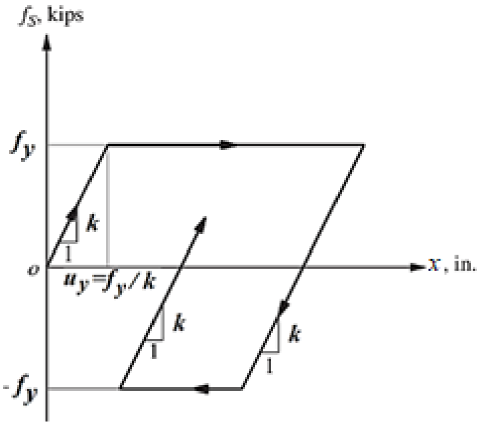


**Fig. 9.** The nonlinear force-deformation behavior of spring in M4 and M5 models.

**Table 11.** Results obtained from LIM and Wilson-θ methods for DOF = 1 of M3 model. The analysis time step is $h$ = 0.02416 $sec$.

| $t_i$ (sec) | Displacement $x_i$ $(in)$ | | Velocity $\dot{x}_i (in/sec)$ | | Acceleration $\ddot{x}_i$ $(in/sec^2)$ | | Error |
|---|---|---|---|---|---|---|---|
| | Wilson-θ | LIM | Wilson-θ | LIM | Wilson-θ | LIM | UF of LIM |
| 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | <1e-5 |
| 0.194 | 0.002 | 0.002 | 0.028 | 0.028 | 0.334 | 0.333 | <1e-5 |
| 0.387 | 0.016 | 0.016 | 0.142 | 0.142 | 0.865 | 0.866 | <1e-5 |
| 0.581 | 0.064 | 0.064 | 0.362 | 0.362 | 1.375 | 1.377 | <1e-5 |
| 0.775 | 0.162 | 0.162 | 0.659 | 0.66 | 1.638 | 1.640 | <1e-5 |
| 0.969 | 0.320 | 0.320 | 0.969 | 0.970 | 1.49 | 1.492 | <1e-5 |
| 1.162 | 0.533 | 0.533 | 1.206 | 1.207 | 0.879 | 0.879 | <1e-5 |
| 1.356 | 0.777 | 0.777 | 1.285 | 1.285 | -0.118 | -0.119 | <1e-5 |
| 1.550 | 1.016 | 1.017 | 1.148 | 1.148 | -1.309 | -1.313 | <1e-5 |
| 1.744 | 1.207 | 1.207 | 0.782 | 0.781 | -2.436 | -2.441 | <1e-5 |
| 1.937 | 1.307 | 1.307 | 0.225 | 0.224 | -3.233 | -3.238 | <1e-5 |
| 2.131 | 1.287 | 1.287 | -0.436 | -0.438 | -3.493 | -3.496 | <1e-5 |
| 2.325 | 1.138 | 1.138 | -1.086 | -1.089 | -3.117 | -3.118 | <1e-5 |
| 2.518 | 0.875 | 0.874 | -1.604 | -1.607 | -2.14 | -2.137 | <1e-5 |
| 2.712 | 0.532 | 0.531 | -1.887 | -1.888 | -0.724 | -0.718 | <1e-5 |
| 2.906 | 0.163 | 0.162 | -1.873 | -1.873 | 0.876 | 0.884 | <1e-5 |
| 3.100 | -0.174 | -0.175 | -1.555 | -1.553 | 2.365 | 2.372 | <1e-5 |
| 3.293 | -0.423 | -0.424 | -0.982 | -0.979 | 3.467 | 3.473 | <1e-5 |
| 3.487 | -0.544 | -0.544 | -0.249 | -0.246 | 3.988 | 3.991 | <1e-5 |
| 3.681 | -0.517 | -0.517 | 0.520 | 0.523 | 3.843 | 3.842 | <1e-5 |
| 3.875 | -0.348 | -0.347 | 1.199 | 1.202 | 3.075 | 3.071 | <1e-5 |
| 4.068 | -0.066 | -0.064 | 1.681 | 1.683 | 1.838 | 1.831 | <1e-5 |
| 4.262 | 0.286 | 0.287 | 1.895 | 1.896 | 0.362 | 0.353 | <1e-5 |
| 4.456 | 0.650 | 0.652 | 1.822 | 1.821 | -1.097 | -1.105 | <1e-5 |
| 4.649 | 0.975 | 0.976 | 1.487 | 1.485 | -2.302 | -2.308 | <1e-5 |
| 4.843 | 1.214 | 1.214 | 0.958 | 0.955 | -3.081 | -3.084 | <1e-5 |

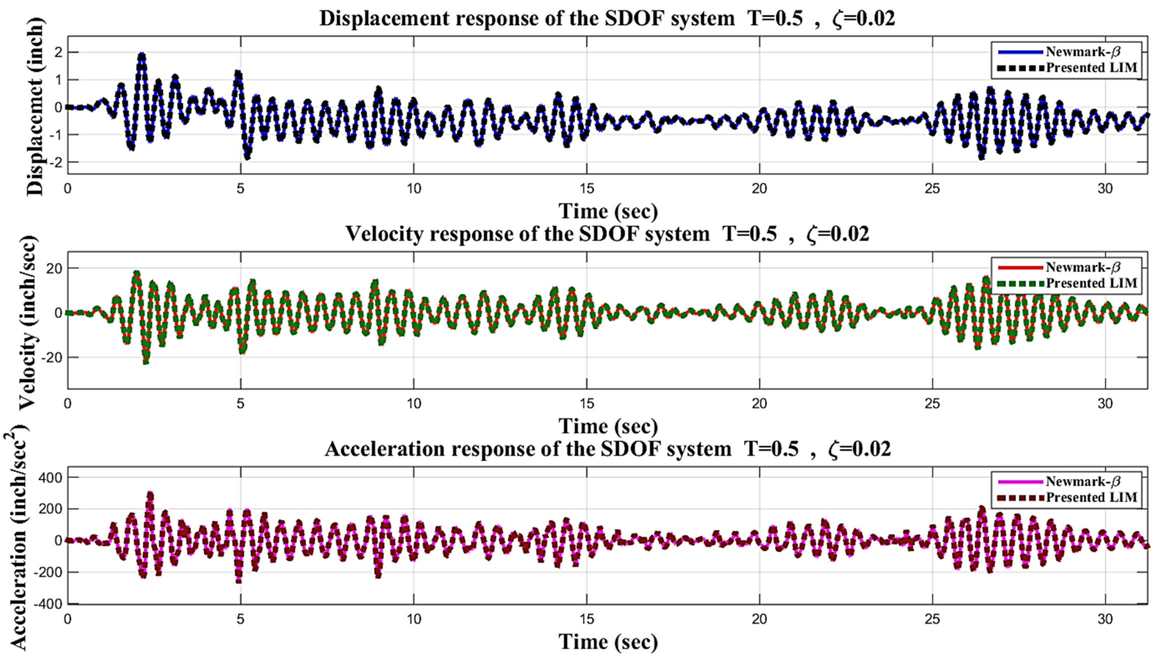


**Fig. 10.** Response time-histories for M3 model under El Centro ground motion record. The analysis time step is $h = 0.002\ sec$.

**Table 12.** Results obtained from LIM and Wilson-θ methods for DOF = 2 of M3 model. The analysis time step is $h = 0.02416$ sec.

| $t_i$ (sec) | Displacement $x_i\ (in)$ | | Velocity $\dot{x}_i\,(in/sec)$ | | Acceleration $\ddot{x}_i\ (in/sec^2)$ | | Error |
|---|---|---|---|---|---|---|---|
| | Wilson-θ | LIM | Wilson-θ | LIM | Wilson-θ | LIM | UF of LIM |
| 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 5.000 | 5.000 | <1e-5 |
| 0.194 | 0.092 | 0.092 | 0.936 | 0.936 | 4.549 | 4.551 | <1e-5 |
| 0.387 | 0.353 | 0.353 | 1.721 | 1.722 | 3.461 | 3.463 | <1e-5 |
| 0.581 | 0.742 | 0.742 | 2.248 | 2.249 | 1.925 | 1.925 | <1e-5 |
| 0.775 | 1.203 | 1.203 | 2.453 | 2.454 | 0.189 | 0.188 | <1e-5 |
| 0.969 | 1.671 | 1.671 | 2.325 | 2.325 | -1.485 | -1.487 | <1e-5 |
| 1.162 | 2.084 | 2.085 | 1.897 | 1.897 | -2.865 | -2.868 | <1e-5 |
| 1.356 | 2.392 | 2.392 | 1.245 | 1.245 | -3.783 | -3.786 | <1e-5 |
| 1.550 | 2.559 | 2.559 | 0.467 | 0.466 | -4.155 | -4.157 | <1e-5 |
| 1.744 | 2.571 | 2.571 | -0.329 | -0.33 | -3.977 | -3.978 | <1e-5 |
| 1.937 | 2.437 | 2.436 | -1.043 | -1.044 | -3.324 | -3.324 | <1e-5 |
| 2.131 | 2.178 | 2.177 | -1.594 | -1.595 | -2.320 | -2.319 | <1e-5 |
| 2.325 | 1.833 | 1.832 | -1.929 | -1.930 | -1.118 | -1.116 | <1e-5 |
| 2.518 | 1.446 | 1.446 | -2.024 | -2.025 | 0.127 | 0.129 | <1e-5 |
| 2.712 | 1.064 | 1.063 | -1.886 | -1.886 | 1.275 | 1.278 | <1e-5 |
| 2.906 | 0.729 | 0.728 | -1.544 | -1.543 | 2.217 | 2.219 | <1e-5 |
| 3.100 | 0.476 | 0.475 | -1.046 | -1.044 | 2.875 | 2.877 | <1e-5 |
| 3.293 | 0.330 | 0.330 | -0.451 | -0.449 | 3.208 | 3.21 | <1e-5 |
| 3.487 | 0.303 | 0.303 | 0.175 | 0.177 | 3.203 | 3.204 | <1e-5 |
| 3.681 | 0.396 | 0.396 | 0.769 | 0.771 | 2.876 | 2.876 | <1e-5 |
| 3.875 | 0.595 | 0.596 | 1.271 | 1.273 | 2.263 | 2.262 | <1e-5 |
| 4.068 | 0.879 | 0.88 | 1.631 | 1.633 | 1.422 | 1.420 | <1e-5 |
| 4.262 | 1.216 | 1.217 | 1.812 | 1.813 | 0.430 | 0.427 | <1e-5 |
| 4.456 | 1.568 | 1.570 | 1.794 | 1.794 | -0.62 | -0.624 | <1e-5 |
| 4.649 | 1.898 | 1.899 | 1.575 | 1.574 | -1.624 | -1.628 | <1e-5 |
| 4.843 | 2.167 | 2.168 | 1.175 | 1.173 | -2.471 | -2.476 | <1e-5 |

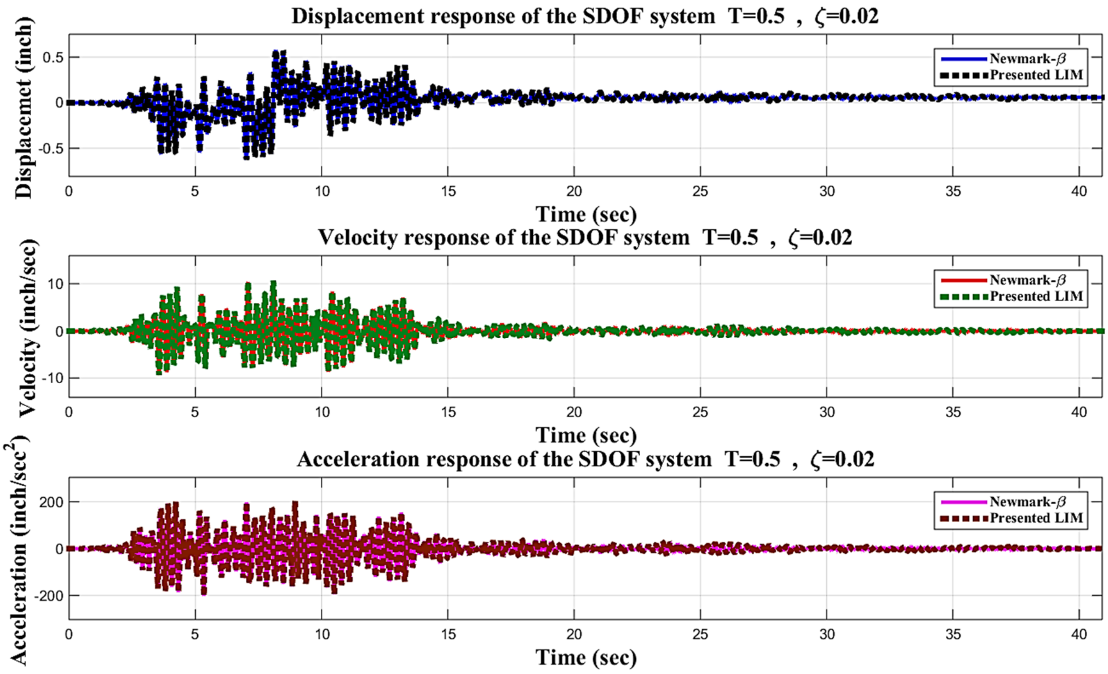


**Fig. 11.** Response time-histories for M3 model under Kobe ground motion record. The analysis time step is $h$ = 0.002 $sec$.

**Table 13**. Summary results obtained from LIM and Newmark-β methods for M4 example model.

| Method | $\ddot{x}_{max}$ (in/sec²) | $\dot{x}_{max}$ (in/sec) | $x_{max}$ (in) | Analysis time (sec) | # of iterations | Time step $h$ (sec) |
|---|---|---|---|---|---|---|
| Newmark-β | 310.84 | 22.773 | 1.9878 | 0.004 | 1 | 0.002 |
| LIM | 310.89 | 22.773 | 1.9879 | 0.007 | 2 | 0.002 |

**Table 14.** Summary results obtained from LIM and Newmark-β methods for M5 example model.

| Method | $\ddot{x}_{max}$ (in/sec²) | $\dot{x}_{max}$ (in/sec) | $x_{max}$ (in) | Analysis time (sec) | # of iterations | Time step $h$ (sec) |
|---|---|---|---|---|---|---|
| Newmark-β | 203.09 | 10.619 | 0.6222 | 0.01 | 1 | 0.001 |
| LIM | 203.09 | 10.619 | 0.6227 | 0.01 | 2 | 0.001 |

**Table 15**. Results obtained from LIM and Newmark-β methods for M4 model under El Centro ground motion record. The analysis time step is $h$ = 0.002$sec$.

| $t_i$ (sec) | Displacement $x_i$ $(in)$ | | Velocity $\dot{x}_i(in/sec)$ | | Acceleration $\ddot{x}_i$ $(in/sec^2)$ | | Error |
|---|---|---|---|---|---|---|---|
| | Newmark-β | LIM | Newmark-β | LIM | Newmark-β | LIM | UF of LIM |
| 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | <1e-5 |
| 1.2 | -0.0850 | -0.0854 | -3.3581 | -3.3675 | -11.8757 | -11.8037 | <1e-5 |
| 2.4 | -1.2078 | -1.2158 | 7.2361 | 7.1670 | 310.8477 | 311.3532 | <1e-5 |
| 3.6 | 0.3915 | 0.3979 | 0.9770 | 1.0389 | -51.2091 | -53.0413 | <1e-5 |
| 4.8 | 0.2059 | 0.1912 | 11.0243 | 10.9513 | 19.3249 | 20.8968 | <1e-5 |
| 6 | -0.1548 | -0.1512 | -9.3317 | -9.3555 | -52.5345 | -54.1921 | <1e-5 |
| 7.2 | -1.2391 | -1.2619 | 0.4257 | 0.4265 | 125.048 | 127.5409 | <1e-5 |
| 8.4 | 0.0119 | 0.0214 | 5.9715 | 6.1226 | -72.9771 | -75.6509 | <1e-5 |
| 9.6 | -0.2991 | -0.3091 | -9.6541 | -9.9965 | -47.1958 | -46.5405 | <1e-5 |
| 10.8 | -0.8388 | -0.8609 | 4.3859 | 4.7024 | 44.1273 | 46.3729 | <1e-5 |
| 12 | -0.8386 | -0.8221 | 8.8008 | 8.7344 | 23.9743 | 20.3033 | <1e-5 |
| 13.2 | -0.2768 | -0.2944 | 1.1746 | 1.0621 | -32.2895 | -30.5482 | <1e-5 |
| 14.4 | -1.3430 | -1.3554 | -4.2643 | -4.1880 | 127.9664 | 128.7883 | <1e-5 |
| 15.6 | -0.3247 | -0.3207 | 2.0107 | 2.0816 | -40.5906 | -42.3419 | <1e-5 |
| 16.8 | -0.2615 | -0.2725 | -2.1303 | -2.2621 | -46.2435 | -45.5329 | <1e-5 |
| 18 | -0.5370 | -0.5480 | 1.1979 | 1.3098 | 4.6152 | 5.1946 | <1e-5 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19.2 | -0.6243 | -0.6259 | -1.9050 | -1.9687 | 8.2134 | 7.4100 | <1e-5 |
| 20.4 | -0.9466 | -0.9617 | 1.9328 | 1.9718 | 94.6731 | 95.9488 | <1e-5 |
| 21.6 | -0.0794 | -0.074 | 5.2212 | 5.2889 | -77.9880 | -79.9605 | <1e-5 |
| 22.8 | -0.8300 | -0.8466 | -2.4453 | -2.6447 | 64.1015 | 65.7323 | <1e-5 |
| 24 | -0.5305 | -0.5376 | 0.4351 | 0.636 | 18.0960 | 18.0233 | <1e-5 |
| 25.2 | 0.0428 | 0.0447 | 4.0032 | 3.902 | -76.4641 | -77.8012 | <1e-5 |
| 26.4 | -1.7896 | -1.8149 | -5.7218 | -5.8211 | 197.2917 | 200.2415 | <1e-5 |
| 27.6 | -0.1297 | -0.1554 | 11.8190 | 12.0148 | -73.9721 | -74.4859 | <1e-5 |
| 28.8 | -0.5220 | -0.544 | -7.7533 | -8.0411 | 17.2394 | 16.3835 | <1e-5 |
| 30 | -0.7012 | -0.7464 | 3.5590 | 3.7654 | 27.3345 | 29.9178 | <1e-5 |
| 31.2 | -0.2935 | -0.3007 | -1.1552 | -1.207 | -36.1145 | -39.417 | <1e-5 |

**Table 16.** Results obtained from LIM and Newmark-β methods for M5 model under Kobe ground motion record. The analysis time step is $h = 0.001$ sec.

| $t_i$ (sec) | Displacement $x_i$ $(in)$ | | Velocity $\dot{x}_i (in/sec)$ | | Acceleration $\ddot{x}_i$ $(in/sec^2)$ | | Error |
|---|---|---|---|---|---|---|---|
| | Newmark-β | LIM | Newmark-β | LIM | Newmark-β | LIM | UF of LIM |
| 0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | <1e-5 |
| 1.2 | -0.0044 | -0.0044 | 0.0634 | 0.0633 | 2.3076 | 2.3075 | <1e-5 |
| 2.4 | 0.0834 | 0.0834 | 0.5092 | 0.5092 | -3.8349 | -3.8365 | <1e-5 |
| 3.6 | -0.4315 | -0.4315 | -6.4858 | -6.4875 | 139.385 | 139.388 | <1e-5 |
| 4.8 | -0.1726 | -0.1726 | -1.0412 | -1.0437 | 65.9732 | 65.9802 | <1e-5 |
| 6.0 | 0.2050 | 0.2050 | 2.2632 | 2.2633 | -63.4683 | -63.4913 | <1e-5 |
| 7.2 | 0.2905 | 0.2905 | -3.2633 | -3.2611 | -115.644 | -115.677 | <1e-5 |
| 8.4 | 0.1663 | 0.1662 | 6.3621 | 6.3616 | 59.2524 | 59.3234 | <1e-5 |
| 9.6 | 0.0067 | 0.0065 | 2.0974 | 2.0969 | 51.0068 | 51.0683 | <1e-5 |
| 10.8 | 0.3825 | 0.3825 | -2.2466 | -2.2443 | -149.500 | -149.509 | <1e-5 |
| 12.0 | -0.1641 | -0.1640 | 2.0364 | 2.0374 | 77.6630 | 77.6630 | <1e-5 |
| 13.2 | -0.0772 | -0.0771 | 6.2321 | 6.2315 | 73.4963 | 73.5237 | <1e-5 |
| 14.4 | 0.1795 | 0.1795 | 0.0834 | 0.0832 | -34.1860 | -34.1664 | <1e-5 |
| 15.6 | 0.0518 | 0.0519 | -0.5109 | -0.5108 | 4.2475 | 4.2473 | <1e-5 |
| 16.8 | -0.0328 | -0.0328 | 0.5014 | 0.5012 | 33.7307 | 33.7296 | <1e-5 |
| 18.0 | 0.0184 | 0.0185 | 1.3043 | 1.3041 | -0.3603 | -0.3544 | <1e-5 |
| 19.2 | 0.0555 | 0.0555 | 0.9388 | 0.9390 | -8.6223 | -8.6150 | <1e-5 |
| 20.4 | 0.0524 | 0.0525 | -0.2661 | -0.2660 | 6.7732 | 6.7764 | <1e-5 |
| 21.6 | 0.0526 | 0.0527 | 0.8213 | 0.8213 | 13.8447 | 13.8459 | <1e-5 |
| 22.8 | 0.0881 | 0.0881 | -0.2277 | -0.2277 | -1.4780 | -1.4786 | <1e-5 |
| 24.0 | 0.0845 | 0.0846 | -0.7259 | -0.7258 | -10.6896 | -10.6927 | <1e-5 |
| 25.2 | 0.0602 | 0.0602 | -0.6220 | -0.6219 | -1.5850 | -1.5882 | <1e-5 |
| 26.4 | 0.0765 | 0.0766 | -0.7176 | -0.7176 | -11.6818 | -11.6873 | <1e-5 |
| 27.6 | 0.0786 | 0.0787 | -0.5572 | -0.5571 | -7.8821 | -7.8856 | <1e-5 |
| 28.8 | 0.0653 | 0.0653 | 0.4149 | 0.4150 | 0.2215 | 0.2218 | <1e-5 |
| 30.0 | 0.0553 | 0.0553 | -0.0547 | -0.0547 | 2.0768 | 2.0767 | <1e-5 |
| 31.2 | 0.0464 | 0.0465 | -0.0643 | -0.0644 | 2.2237 | 2.2231 | <1e-5 |
| 32.4 | 0.0800 | 0.0801 | -0.1641 | -0.1641 | -10.4751 | -10.4761 | <1e-5 |
| 33.6 | 0.0459 | 0.0460 | -0.1046 | -0.1047 | 5.4768 | 5.4750 | <1e-5 |
| 34.8 | 0.0754 | 0.0754 | 0.1409 | 0.1409 | -4.6664 | -4.6662 | <1e-5 |
| 36.0 | 0.0436 | 0.0436 | 0.0779 | 0.0779 | 3.0335 | 3.0344 | <1e-5 |
| 37.2 | 0.0340 | 0.0341 | 0.2269 | 0.2269 | 9.3900 | 9.3908 | <1e-5 |
| 38.4 | 0.0509 | 0.0510 | 0.1142 | 0.1142 | 3.6368 | 3.6376 | <1e-5 |
| 40.8 | 0.0571 | 0.0572 | 0.0357 | 0.0357 | 0.6898 | 0.6903 | <1e-5 |

A detailed comparison of results has also been provided in Tables 15 and 16 for M4 and M5 models, respectively. The analysis results clearly show that there is a very good agreement between the responses obtained from LIM with that from nonlinear version of Newmark-β method for both cases. It is noted that the nonlinear LIM is easy to use rather than the others. In spite of linear LIM, nonlinear LIM is applicable to both damped and undamped structures. Hence, analyzing the undammed linear systems, which are not permitted by the linear LIM, can be perfectly treated by this algorithm

## 4. Conclusions

In this study, a novel numerical method, so-called load impulse method (LIM), was proposed for estimating the dynamic response of the linear and nonlinear SDOF and MDOF systems. LIM is based on the simple concepts from dynamics and applied mathematics. This method estimates the kinematic response of the dynamic systems using a manipulated form of DEOM. The accuracy and efficiency of the proposed LIM was explored for several linear and nonlinear SDOF systems and a linear MDOF case. The analysis results clearly showed that the proposed method can satisfactorily estimate the displacement, velocity, and acceleration response of dynamic systems within almost optimal computational effort .

A good agreement between the solution from LIM and conventional methods such as Duhamel, Newmark-β, and Wilson-θ provide evidence that the proposed method is efficient and accurate enough. The differences between the results obtained from LIM and other methods were almost insignificant. Hence, the proposed LIM procedure can be identified as a reliable analysis tool for estimating the dynamic response of structural systems. In summary, the main advantages of the proposed LIM are: (1) it can be generally used for estimating the dynamic response of linear and nonlinear SDOF models, as well as MDOF systems subjected to any arbitrary dynamic load. (2) It has a user-friendly programming pattern. (3) The estimation errors from the proposed method are not accumulated in the next instances. Further studies on the development of the method for estimating the seismic response of undamped linear and nonlinear MDOF is still underway.

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