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Numerical solution of optimal control problem for economic growth model using RBF collocation method

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Abstract

In the current paper, for the economic growth model, an efficient numerical approach on arbitrary collocation points is described according to Radial Basis Functions (RBFs) interpolation to approximate the solutions of optimal control problems. The proposed method is based on parametrizing the solutions with any arbitrary global RBF and transforming the optimal control problem into a constrained optimization problem using arbitrary collocation points. The superiority of the method is its flexibility to select between different RBF functions for the interpolation and also parametrization an extensive range of arbitrary nodes. The Lagrange multipliers method is employed to convert the constrained optimization problem into a system of algebraic equations. Numerical results approve the accuracy and performance of the presented method for solving optimal control problems in the economic growth model.

Keywords. Optimal control problem, Economic growth model ,RBF collocation method, Lagrange multipliers.

2010 Mathematics Subject Classification. 49-XX, 91B62 , 65L60, 70H03.

1. Introduction

Economic growth is considered as an inseparable part and one of the most significant characteristics of economic development. Economic growth models can be used for planning, analysis and forecasting relationships among global economic indicators, which include production facilities, labor force, national income, etc. Various aspects of economic growth modeling are introduced by Grygorkiv et al. [8], Novales et al.[13], Solow [17] , etc.

Radial basis functions (RBFs) are traditional and powerful tools for multivariate scattered data interpolation [9]. The radial basis function technique was first applied by Hardy in 1971 in connection with a topological application on quadric surfaces and later used for the partial differential equations (PDEs) solution by [10]. Recently, by using the collocation method, the researchers combined Legendre-Gauss-Lobatto (LGL) nodes with RBFs for the approximation of the solution [15]. Golbabai et al.[7] used RBFs for solving variational problems.

In the current paper, we propose an RBF collocation method to solve the optimal control problem for the economic growth model. In a direct method, the states and/or controls are approximated by a specific function with unknown coefficients, and the optimal control problem is discretized using a set of proper nodes (collocation points) to eventually transcribe it into a constrained optimization problem. The interpolation function can be selected from the global RBFs family which includes Gaussian (GA), multiquadric (MQ), inverse multiquadric (IMQ), etc.

An attractive feature of the RBF method is the arbitrary discretization. This fact that our presented approach does not need an special grid of nodes for the discretization, which makes it more applicable. To approximate integration, we use the Legendre-Gauss-Lobatto quadrature. Using the method of Lagrange multipliers, the problem is transformed to an algebraic equations system. The unknown coefficients are determined by solving algebraic equations.

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This paper is structured as follows: We illustrate the optimal control problem for the economic growth model in section 2. In section 3, the radial basis functions properties are described. In section 4, we implement the optimal control problem with the collocation method according to radial basis functions. Numerical examples are presented in section 5 to indicate the applicability and the accuracy of the proposed method. A conclusion of this work is given in the final section.

2. Economic Growth Model

At first, we consider a mathematical model of a highly simplified economy in which the rate of output, $Y(t)$, is assumed to depend on the rates of input of capital, $K(t)$ (for example in the form of machinery) and labour force, $L(t)$; that is

$$Y = F(K, L), \tag{2.1}$$

where F is called the production function. This function is assumed to obey a "return to scale" property so that

$$F(\alpha K, \alpha L) = \alpha F(K, L), \tag{2.2}$$

with $\alpha = \frac{1}{L}$, and defining the output per worker, $y = \frac{Y}{L}$, and capital per worker, $k = \frac{K}{L}$, we have

$$y = \frac{Y}{L} = \frac{F(K, L)}{L} = F(K, 1) = f(k), \tag{2.3}$$

the function $f(k)$ will satisfy the conditions

$$f(0) = 0, f'(k) > 0, f''(k) \leq 0, \lim_{k\to\infty} f(k) = \infty. \tag{2.4}$$

Output is either consumed or invested, so that

$$Y(t) = C(t) + I(t), \tag{2.5}$$

where C and I are the rates of consumption and investment, respectively. The investment is used to increase the capital stock and replace machinery, that is,

$$I = \frac{dK}{t} + \mu K, \tag{2.6}$$

here μ is called the rate of depreciation. Defining $c = \frac{C}{L}$ as the consumption rate per worker, we see that

$$y = f(k) = c(t) + \frac{1}{L}\frac{dK}{dt} + \mu k, \tag{2.7}$$

and since

$$\frac{d}{dt}\Big(\frac{K}{L}\Big) = \frac{1}{L}\frac{dK}{dt} - \frac{K}{L^2}\frac{dL}{dt} = \frac{1}{L}\frac{dK}{dt} - \frac{k}{L}\frac{dL}{dt}, \tag{2.8}$$

we have

$$y = f(k) = c(t) + \frac{dk}{dt} + \frac{k}{L}\frac{dL}{dt} + \mu k. \tag{2.9}$$

Assuming that labour grows exponentially, i.e. $L = L_0 e^{\rho t}$, so

TABLE 1. Some popular RBFs ($r = \|(x, y) - (x_i, y_j)\|, \rho > 0.$)

Name	$\phi(r)$
Gaussian (GA)	$\phi(r) = e^{-\rho r^2}$
Multiquadric (MQ)	$\phi(r) = (\rho^2 + r^2)^{1/2}$
Inverse quadratic(IQ)	$\phi(r) = \frac{1}{r^2+\rho^2}$
Inverse multiquadric (IMQ)	$\phi(r) = (\rho^2 + r^2)^{-1/2}$

$$\frac{dk}{dt} = f(k) - (\rho + \mu)k - c(t), \tag{2.10}$$

which is the governing equation of this economic growth model [2]. The consumption rate per worker, $c(t)$, is the control for this problem. The central planner's problem is to choose c on a time interval $0 \le t \le T$ in some best way. But what are the desired economic objectives? One method of quantifying the best way is to introduce a 'utility' function $U = U(c)$, which is a measure of the value attached to consumption. The function U will normally satisfy

$$U(c) > 0, U'(c) > 0, U''(c) < 0, \lim_{c\to 0} U'(c) = \infty, \lim_{c\to\infty} U'(c) = 0, \tag{2.11}$$

which means that a fixed increment in consumption will be valued increasingly highly with decreasing consumption level. We also need to optimize consumption for all time in $[0, T]$ but with some discounting for future time. So the central planner wishes to maximize the "welfare" integral

$$W = \int_0^T e^{-\delta t} U(c(t))dt, \tag{2.12}$$

where δ is known as the discount rate, which is a measure of preference for earlier rather than later consumption. If $\delta = 0$, then there is no time discounting, and consumption is valued equally at all times; as δ increases, so does the discounting of consumption and utility at future times. The mathematical problem has now been reduced to finding the optimal consumption path $c(t), 0 \le t \le T$, which maximizes W subject to the constraint (2.10), and, for example, with $k(0) = k_0, k(T) = k_T$. The problem is to choose the consumption path $c(i), 0 \le t \le T$ which maximizes the welfare integral of the following form:

$$W = \int_0^T e^{-\delta t} U(c(t))dt, \tag{2.13}$$

while satisfying the growth equation

$$\frac{dk(t)}{dt} = f(k(t)) - (\rho + \mu)k(t) - c(t), \tag{2.14}$$

$$k(0) = k_0, k(T) = k_T. \tag{2.15}$$

3. Definition of the RBFs

First, RBFs were studied for approximating functions by Hardy[9]. This technique allowed scattered data to be simply applied in computations. The RBFs contain two useful specifications: the existence of a free positive parameter known as the shape parameter [6] and set of scattered centers with the possibility of choosing their positions. Some popular RBFs with shape parameter ρ are listed in Table 1 and are plotted in Figure 1 .

Where r would be the Euclidean distance between a fixed point $x \in R^d$ and any $y \in R^d$ i.e. $\|x - y\|$ and ρ is the shape parameter[1, 5]. Notice that the accuracy of RBF-based methods highly depends upon the shape parameter ρ of the basis functions, which is responsible for the flatness of such functions. Indeed, the shape of the RBF is becoming flatter the condition number of the system is also growing and the approximation is more accurate. This behavior as uncertainty principle [16] is manifested referring to the fact that the accuracy and well conditioned cannot occur at the same time for an RBF approximant. In general, let $x_1, ..., x_N$ be a presented set of distinct points in R^d. The main idea is using RBF interpolation by translation of a single function i.e. the interpolating RBFs approximation is considered as

$$y(x) \cong y^N(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x), \tag{3.1}$$

where $\phi_i(x) = \phi(\|x - x_i\|)$, x_i would be centers and α_i are considered unknown coefficients for $i = 1, ...N$. Consider that the given values $f_i = f(x_i), i = 1, ..., N$ must be interpolated. The unknown coefficients $\{\alpha_i\}_{i=1}^N$ will be calculated in a way that $y(x_j) = f_j$ for $j = 1, ..., N$ which results in the following linear system of equations

$$AX = f, \tag{3.2}$$

where

$$X = [x_1, ..., x_n], f = [f_1, ..., f_n],$$

$$\mathbf{A} = \begin{bmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_1 - x_2\|) & \dots & \phi(\|x_1 - x_N\|) \\ \phi(\|x_2 - x_1\|) & \phi(\|x_2 - x_2\|) & \dots & \phi(\|x_2 - x_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|x_N - x_1\|) & \phi(\|x_N - x_2\|) & \dots & \phi(\|x_N - x_N\|) \end{bmatrix}.$$

If ϕ_i has global support, this technique would produce a dense matrix A. Using Schoenberg's theorem [4], the matrix A is shown to be positive definite (consequently nonsingular) for distinct interpolation points for GA, IMQ and IQ. Also using the Micchelli theorem [11], we can show that A is invertible for distinct sets of the scattered points in the case of MQ. For solving optimal control problems, the RBF collocation approach would be according to the interpolating global RBFs on arbitrary collocation points. In order to provide a more flexible framework, including unequally and equally spaced nodes, different sets of collocation points could be arbitrarily selected for discretization. For instance, to discretize the problem, a set of Gauss-Legendre(GL), Gauss-Laguerre, Chebyshev-Gauss (CG), Gauss-Lobatto, Chebyshev-Gauss-Lobatto (CGL) and Gauss-Legendre-Lobatto (GLL) nodes, could be chosen as a set of unequally-spaced orthogonal nodes [12, 15].

4. The proposed method

Now we aim to apply the RBFs method for solving the optimal control of Eqs. (1)-(3). The state $k(t)$ and control $c(t)$ are approximated using N RBFs as

$$k(t) \simeq k^N(t) = \sum_{i=1}^{N} \alpha_i \phi(\|t - t_i\|) = \sum_{i=1}^{N} \alpha_i \phi_i(t), \tag{4.1}$$

$$c(t) \simeq c^N(t) = \sum_{i=1}^{N} \beta_i \phi(\|t - t_i\|) = \sum_{i=1}^{N} \beta_i \phi_i(t), \tag{4.2}$$

where $k^N(t)$ and $c^N(t)$ denote the RBF interpolation of $k(t)$ and $c(t)$, respectively. Also, $\phi_i(t)$ is the RBF and α_i, β_i are RBF weights related to $k^N(t)$, $c^N(t)$, respectively. Differentiating the expression in Eq. (4.1) with respect to t yields

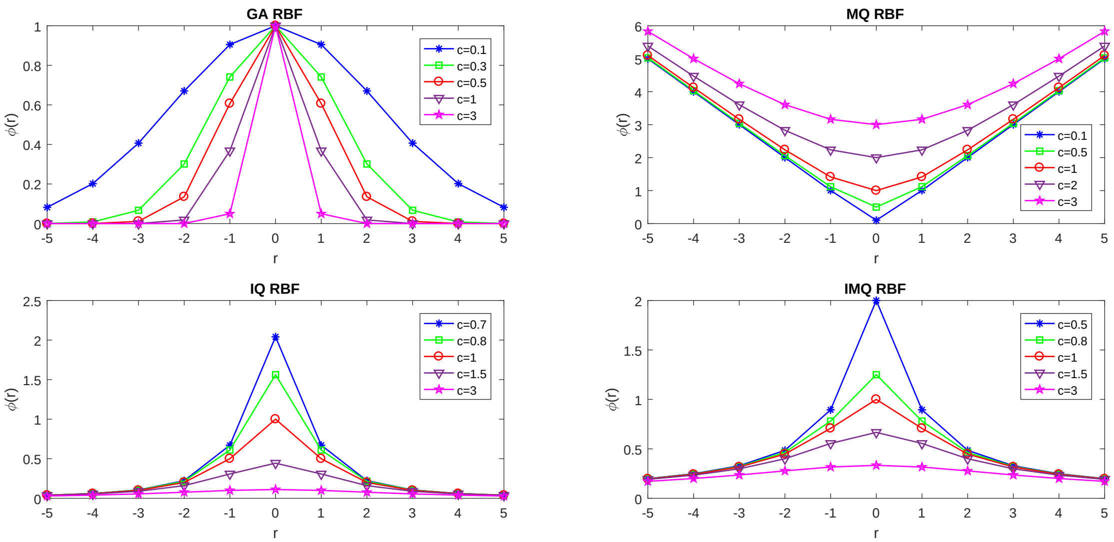

FIGURE 1. Figures of some well-known RBFs.

$$\dot{k(t)} \simeq \dot{k}^N(t) = \sum_{i=1}^{N} \alpha_i \dot{\phi}(\|t - t_i\|) = \sum_{i=1}^{N} \alpha_i \dot{\phi}_i(t). \tag{4.3}$$

Remark 4.1. Let $k^N(t) = \sum\limits_{i=1}^{N} \alpha_i \phi_i(t)$ and $\phi_i(x) = exp(-c\|x - x_i\|^2)$, for $i = 1, ..., N$. Then there exist derivative with respect to t as follows:

$$\dot{k}^N(t) = \frac{d}{dt} k^N(t) = \sum_{i=1}^{N} \alpha_i \frac{d}{dt} \phi_i(t) = \sum_{i=1}^{N} -2c\alpha_i (t - t_i)\phi_i(t). \tag{4.4}$$

Remark 4.2. Let $k^N(t) = \sum\limits_{i=1}^{N} \alpha_i \phi_i(t)$ and $\phi_i(t) = \sqrt{c^2 + \|t - t_i\|^2}$, for $i = 1, ..., N$. Then there exist derivative with respect to t as follows:

$$\dot{k}^N(t) = \frac{d}{dt} k^N(t) == \sum_{i=1}^{N} \alpha_i \frac{d}{dt} \phi_i(t) = \sum_{i=1}^{N} \frac{\alpha_i (t - t_i)}{\sqrt{c^2 + (t - t_i)^2}}. \tag{4.5}$$

Now by substituting remark 1 or 2 in Eq. (2.14) we have

$$\sum_{i=1}^{N} \alpha_i \frac{d}{dt} \phi_i(t) = f(\sum_{i=1}^{N} \alpha_i \phi_i(t)) - (\rho + \mu) \sum_{i=1}^{N} \alpha_i \phi_i(t) - \sum_{i=1}^{N} \beta_i \phi_i(t). \tag{4.6}$$

Applying Gauss-Lobatto, we can approximate the Eq. (2.13) as follows,

$$\int_0^T e^{-\delta t} U(c(t)) dt = \frac{T}{2} \sum_{k=1}^{M} w_k e^{-\delta t} U\Big(\sum_{i=1}^{N} \beta_i \phi_i(t_k) \Big), \tag{4.7}$$

where $t_k = \frac{T}{2}s_k + \frac{T}{2}$, s_ks are nodes LGL and w_ks are LGL weights corresponding to LGL nodes $s_k \in [-1, 1]$, given by

$$w_k = \frac{2}{N(N-1)\Big(P_{N-1}(s_k)\Big)^2} \qquad k = 1, ..., M, \tag{4.8}$$

here P_{N-1} is the Legendre polynomial of degree $N-1$, also $t_i, i = 1, ..., N$ are RBF centers nodes which selected arbitrary way.
Finally, the optimal control problem Eqs. (2.13)-(2.14) will be decreased to a constrained optimization problem

$$J(a) = \frac{T}{2}\sum_{k=1}^{M} w_k e^{-\delta t} U\Big(\sum_{i=1}^{N} \beta_i \phi_i(t_k)\Big), \tag{4.9}$$

subject to

$$\sum_{i=1}^{N} \alpha_i \frac{d}{dt}\phi_i(t) = f(\sum_{i=1}^{N} \alpha_i \phi_i(t)) - (\rho + \mu)\sum_{i=1}^{N} \alpha_i \phi_i(t) - \sum_{i=1}^{N} \beta_i \phi_i(t), \tag{4.10}$$

$$\sum_{i=1}^{N} \alpha_i \phi_i(0) = k_0, \sum_{i=1}^{N} \alpha_i \phi_i(T) = k_T. \tag{4.11}$$

For solving the optimization problem of Eqs. (4.9)-(4.11), we use the Lagrange multipliers method and convert the constrained optimization problem into an unconstrained optimization problem

$$J^*(a) = J(a) + \lambda_1\Bigg(\sum_{i=1}^{N} \alpha_i \frac{d}{dt}\phi_i(t) - f(\sum_{i=1}^{N} \alpha_i \phi_i(t)) + (\rho + \mu)\sum_{i=1}^{N} \alpha_i \phi_i(t) + \sum_{i=1}^{N} \beta_i \phi_i(t)\Bigg)$$
$$+\lambda_2\Bigg(\sum_{i=1}^{N} \alpha_i \phi_i(0) - k_0\Bigg) + \lambda_3\Bigg(\sum_{i=1}^{N} \alpha_i \phi_i(T) - k_T\Bigg), \tag{4.12}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the Lagrange multipliers. The coefficients α_i, β_i, for $i = 1, ..., N$ will be obtained after solving the following algebraic system.

$$\frac{\partial J^*(a)}{\partial \alpha_i} = 0, \frac{\partial J^*(a)}{\partial \beta_i} = 0, i = 1, ..., N, \tag{4.13}$$

$$\frac{\partial J^*(a)}{\partial \lambda_i} = 0, i = 1, 2, 3. \tag{4.14}$$

5. Numerical results

To test the efficiency of the proposed method on the optimal control problem for the economic growth model, we present an example. We use the utility function

$$U(c(t)) = \frac{c(t)^{1-\nu}}{1-\nu}, \tag{5.1}$$

where $\nu > 0$, so that marginal utility has the constant elasticity $-\nu$. For comparison of accuracy, the following two types of errors are used:

- Maximum absolute error:

$$e_\infty = max|k_i - \bar{k}_i|, \quad for\ i = 1, ..., N. \tag{5.2}$$

- Root mean square error:

$$e_{RMS} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |k_i - \bar{k}_i|^2}, \tag{5.3}$$

where k_i denotes the exact solution and $\bar{k}_i$ denotes the numerical solution which is obtained employing proposed methods. The condition number of a square matrix W is defined as

$$\kappa(W) = \|W\|_2 \|W^{-1}\|_2. \tag{5.4}$$

The stability is studied by evaluating the condition number (CN) obtained by using the Matlab command condest. If we suppose $\nu = 0.5$ and $f(k) = k$ another way of formulating the problem is as follows: Find the optimal consumption path $c = c(t)$, with $k(0) = k_0$ that maximizes

$$\int_0^T 2e^{-\delta t}\sqrt{c(t)}dt, \tag{5.5}$$

subject to

$$\frac{dk(t)}{dt} = (1 - \rho + \mu)k(t) - c(t), \tag{5.6}$$

$$k(0) = k_0, k(T) = k_T, \tag{5.7}$$

if we assume $\rho = \mu = 0.5, \delta = 1, k(0) = 2$ and $k(T) = 10$ and let the period to be $T = 5$ years, using RBF method with $c = 1$ and $N = 25$, the solution will be obtained as in Figure 2.

The accuracy of the RBF collocation method is clearly illustrated in graphs even with the least number of points (i.e., $N = 25$). These figures show that the approximated numerical solutions are matched with the corresponding exact solution. We solved the algebraic system by the QR method. Figure 3 demonstrates the superiority of GA-RBF to MQ-RBF in the condition number criterion for different sets of collocation points. The authors designed another numerical experiment to investigate the effect of center point choosing on the accuracy of the solution. They considered uniform, Gauss-Lobatto, Gauss-Chebyshev and Gauss-Kronrod center points for RBF. In Tables 2 and 3, for some values of the shape parameter ρ, comparison of Maximum absolute errors (e_∞) and RMS error (e_{RMS}) calculated over the uniform, Gauss-Lobatto, Gauss-Chebyshev and Gauss-Kronrod center points for N = 15 and N = 45 global data points. Comparison of RMS and maximum point-wise errors in Table 2 and 3 illustrate that the numerical solution obtained with MQRBF method are more accurate than GARBF method. As it can be seen in Tables 2 and 3 , the accuracy of numerical solutions for Gauss-Lobatto distribution of points are significantly better than other distribution of points. Numerical results verify that for Values close to one of shape parameter for instance ρ =0.8 and 1.2 the MQ-RBF is significantly suitable than GA-RBF method. In Table 4 we compare the results obtained of our method with other methods applied in [3, 14].

TABLE 2. Comparison of maximum absolute errors and RMS error calculated over the uniform, Gauss-Lobatto, Gauss-Chebyshev and Gauss-Kronrod center points for capital (k).

	Uniform points		lobatto points		Chebyshev points		Kronrod points	
	15	45	15	45	15	45	15	45
c=0.4								
e_∞ for GARBF	1.72e-2	1.62e-3	5.03e-3	8.82e-4	4.32e-4	1.71e-1	5.26e-3	1.42e-1
e_∞ for MQRBF	1.17e-2	2.31e-3	3.46e-3	1.47e-3	3.90e-3	4.32e-3	4.02e-3	1.71e-3
e_{RMS} for GARBF	8.30e-3	5.73e-4	1.94e-3	4.31e-4	1.61e-4	6.94e-2	2.08e-3	6.08e-2
e_{RMS} for MQRBF	5.03e-3	8.25e-4	1.27e-3	6.80e-4	1.57e-3	1.64e-3	1.87e-3	6.78e-4
c=0.8								
e_∞ for GARBF	1.17e-2	3.34e-2	1.39e-2	5.28e-3	5.69e-2	5.87e-3	8.14e-2	6.44e-3
e_∞ for MQRBF	4.21e-3	1.09e-3	1.51e-3	4.97e-4	1.55e-3	5.69e-4	1.61e-3	8.55e-4
e_{RMS} for GARBF	3.90e-3	2.02e-2	7.53e-3	2.44e-3	2.04e-2	2.60e-3	2.73e-2	2.79e-3
e_{RMS} for MQRBF	2.19e-3	4.64e-4	6.69e-4	1.72e-4	7.86e-4	2.22e-4	8.54e-4	4.55e-4
c=1.2								
e_∞ for GARBF	1.54e-2	2.81e-1	2.23e-2	5.70e-2	7.04e-2	2.42e-2	1.19e-1	3.50e-2
e_∞ for MQRBF	2.61e-3	1.85e-2	6.30e-4	2.26e-4	8.75e-4	2.62e-3	8.95e-4	8.12e-4
e_{RMS} for GARBF	7.14e-3	7.83e-2	6.82e-3	1.86e-2	2.34e-2	8.69e-3	4.22e-2	1.24e-2
e_{RMS} for MQRBF	1.56e-3	6.76e-3	3.15e-4	1.07e-4	3.87e-4	1.02e-3	4.02e-4	3.62e-4

TABLE 3. Comparison of maximum absolute errors and RMS error calculated over the uniform, Gauss-Lobatto, Gauss-Chebyshev and Gauss-Kronrod center points for consumption (c).

	Uniform points		lobatto points		Chebyshev points		Kronrod points	
	15	45	15	45	15	45	15	45
c=0.4								
e_∞ for GARBF	3.79e-1	4.19e-2	1.05e-1	1.72e-2	9.58e-3	2.84	8.15e-2	2.21
e_∞ for MQRBF	6.02e-1	1.29e-1	1.58e-1	8.67e-2	1.60e-1	1.10e-1	1.41e-1	8.48e-2
e_{RMS} for GARBF	1.13e-1	7.36e-3	2.98e-2	4.85e-3	2.87e-3	7.57e-1	2.40e-2	6.18e-1
e_{RMS} for MQRBF	2.11e-1	2.58e-2	5.54e-2	2.41e-2	6.11e-2	3.26e-2	5.67e-2	1.94e-2
c=0.8								
e_∞ for GARBF	4.69e-1	4.71e-1	3.16e-1	1.21e-1	7.44e-1	1.49e-1	1.00	1.19e-1
e_∞ for MQRBF	2.56e-1	4.38e-2	5.73e-2	4.88e-3	5.54e-2	2.22e-2	4.90e-2	4.00e-2
e_{RMS} for GARBF	1.29e-1	1.55e-1	9.70e-2	3.21e-2	2.20e-1	3.61e-2	2.93e-1	3.26e-2
e_{RMS} for MQRBF	7.88e-2	9.71e-3	1.73e-2	1.92e-3	1.78e-2	4.90e-3	1.63e-2	1.01e-2
c=1.2								
e_∞ for GARBF	4.42e-1	4.23	5.28e-1	9.82e-1	3.67e-1	1.89e-1	6.19e-1	2.07e-1
e_∞ for MQRBF	7.68e-2	1.20e-1	2.38e-2	6.90e-3	1.02e-2	3.35e-2	1.09e-2	1.51e-2
e_{RMS} for GARBF	1.32e-1	7.88e-1	1.51e-1	2.55e-1	2.42e-1	7.42e-2	3.50e-1	8.85e-2
e_{RMS} for MQRBF	2.38e-2	5.07e-2	6.64e-3	1.70e-3	5.75e-3	1.23e-2	5.97e-3	4.77e-3

TABLE 4. Comparison between errors of the present method with other methods.

Method	capital (k)		consumption (c)	
	e_∞	e_{RMS}	e_∞	e_{RMS}
Neural network [3]	1.07e-3	7.82e-4	1.47e-1	6.90e-2
Euler and Fuzzy Systems [14]	4.07e-5	1.37e-5	2.26e-2	6.90e-3
Proposed approach	2.38e-5	5.89e-6	6.57e-5	2.14e-5

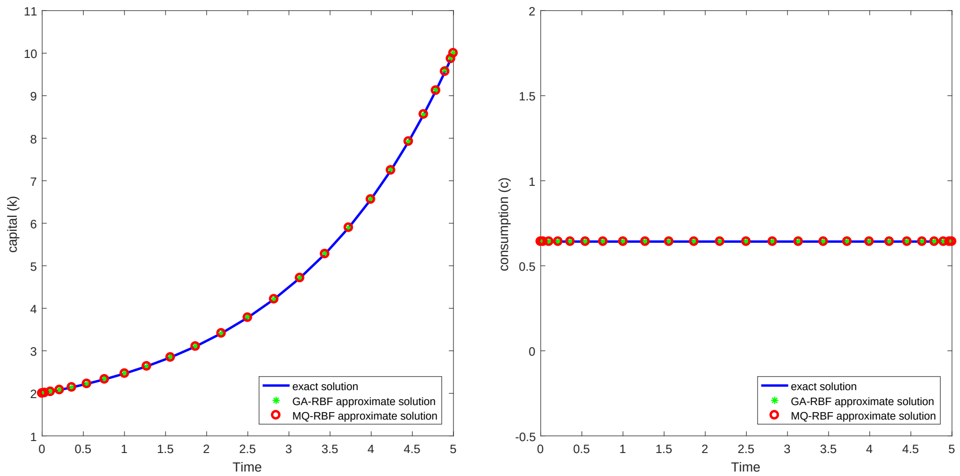

FIGURE 2. Solution to the optimal control Problem using GA-RBF and MQ RBF for N = 25 (a) capital (k), (b) consumption (c).

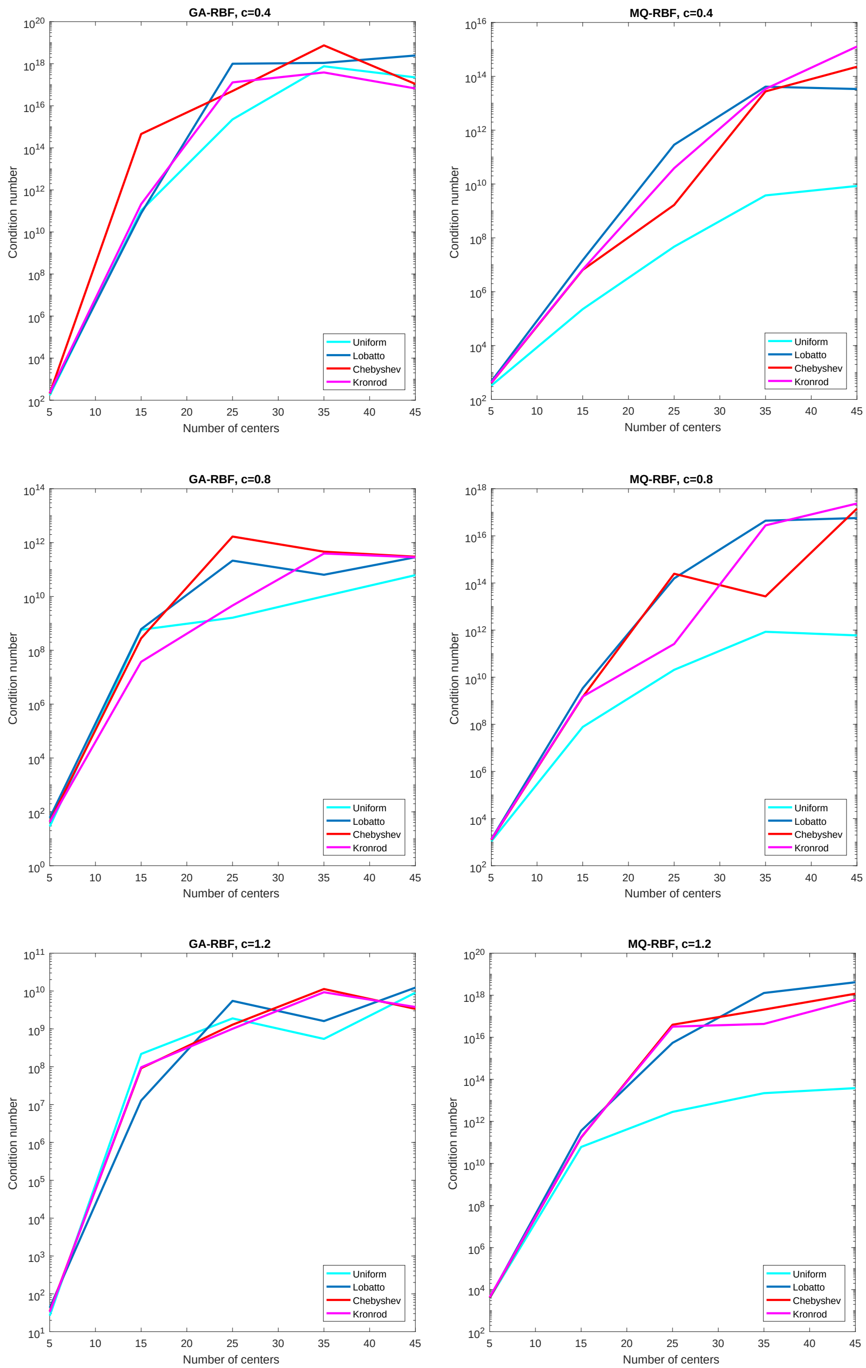

FIGURE 3. Behaviour of condition number when number of centers (N) increases for uniform, Gauss-Lobatto, Gauss-Chebyshev and Gauss-Kronrod centers points.

6. Conclusion

In this paper, we have investigated the application of RBF interpolating in optimal control problem for the economic growth model. The main goal of this paper was to introduce an RBF collocation method to approximate the solution of optimal control problem which is meshfree characteristic and does not require mesh generation. The meshfree nature of the new technique allows us to solve problems with non-regular geometry. Combining the RBF collocation method with Legendre-Gauss-Lobatto quadrature, the optimal control problem decreases to an algebraic system. The numerical results confirmed the efficiency of our proposed method for solving optimal control problem for the economic growth model.

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