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On the numerical treatment and analysis of Hammerstein integral equation

Maryam Derakhshan
Department of Mathematics,
University of Mohaghegh Ardabili,56199-11367, Ardabil, Iran.
E-mail: m.derakhshan@uma.ac.ir

Mohammad Zarebnia*
Department of Mathematics,
University of Mohaghegh Ardabili,56199-11367, Ardabil, Iran.
E-mail: zarebnia@uma.ac.ir

Abstract In this paper, we study the quadratic rules for the numerical solution of Hammerstein integral equation based on spline quasi-interpolant. Also the convergence analysis of the methods are given. The theoretical behavior is tested on examples and it is shown that the numerical results confirm theoretical part.

Keywords. Spline, Quasi-interpolant, Quadrature, Hammerstein, Convergence.
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1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics, physics and engineering problems. Nonlinear integral equations have been considered in relation to vehicular traffic, biology, the theory of optimal control, economics, etc for example, see [4]. Some numerical methods for approximating the solution of Hammerstein integral equations are known [5, 7, 8, 13]. For Hammerstein integral equations, the classical method of successive approximations was defined in [19]. A variation of the Nyströms method was proposed in [8]. A collection method was introduced in [7]. The methods in [7] transform a given integral equation into a system of nonlinear equations, which has to be solved with some manners of iterative methods. Moreover, one can refer to other methods such as [1, 2, 11, 12]. We consider following nonlinear integral equation of Hammerstein type

$$u(x) - \int_a^b k(x,t)\mathfrak{F}(t,u(t))dt = g(x), \tag{1.1}$$

where g and k are known continuous functions, with $\mathfrak{F}(t,u(t))$ nonlinear in u and u is the unknown function that to be determined.
Integration of a function on bounded interval or on a certain district is an important operation for many physical problems. In this paper, we study a new class of endpoint

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∗ corresponding.

corrected rules based on integrating spline quasi-interpolant and using only function values inside the interval of integration.
A discrete quasi-interpolant (abbr. dQI) of degree d is a spline operator of the form:

$$\mathbb{P}_d f = \sum_{i \in J} \zeta_i(f) B_i,$$

whose coefficients $\zeta_i(f)$ are linear combinations of values of f on either the set T_n (for d even) or on the set X_n (for d odd), where

$$t_n = \{t_j = \frac{x_{j-2} + x_{j-1}}{2}, j \in \{1, 2, \cdots, n+2\}\},$$
$$X_n = \{x_j = a + jh, 0 \leq j \leq n\}.$$

The organization of the paper is as follows. In Section 2, we describe the construction of quadrature rules of arbitrary low and high orders based on quadratic spline quasi-interpolant. In Section 3, we give an application of the quadrature rules of Section 2 to the numerical solution of Hammerstein integral equations. In Section 4, the convergence and error analysis of the numerical solution are provided. At the end we give some numerical examples which confirm our theoretical results.
In this paper, We consider the following quadratic spline quasi-interpolant by

$$\mathbb{P}_2 f = \sum_{i \in J} \zeta_i(f) B_i, \tag{1.2}$$

where

$$\zeta_1(f) = f_1, \tag{1.3}$$
$$\zeta_{n+2}(f) = f_{n+2}, \tag{1.4}$$
$$\zeta_2(f) = -1/3 f_1 + 3/2 f_2 - 1/6 f_3 = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, \tag{1.5}$$
$$\zeta_{n+1}(f) = -1/6 f_n + 3/2 f_{n+1} - 1/3 f_{n+2} = \beta_3 f_n + \beta_2 f_{n+1} + \beta_1 f_{n+2}, \tag{1.6}$$

and for $3 \leq j \leq n$,

$$\zeta_j(f) = -1/8 f_{j-1} + 5/4 f_j - 1/8 f_{j+1} = \gamma_1 f_{j-1} + \gamma_2 f_j + \gamma_3 f_{j+1}, \tag{1.7}$$

with $f_i = f(t_i)$, $t_1 = a$, $t_{n+2} = b$, $t_i = a + (i - 3/2)h$, $2 \leq i \leq n+1$. We study a new class of endpoint corrected rule based on integrating spline quasi-interpolant and using only function values inside the interval of integration. we consider quadrature rule based on this type of spline quasi-interpolant of convergence order $O(h^r), r \geq 4$. For $j = 2, \ldots, n+1$, we appoint the same values of $\mu_j(f)$, given by (1.5), (1.6), (1.7) and for $j = 1$ and $j = n+2$,

$$\mu_1(f) = \sum_{i=1}^{m} \alpha_i f_i, \tag{1.8}$$

$$\mu_{n+2}(f) = \sum_{i=1}^{m} \alpha_i f_{n+3-i}, \tag{1.9}$$

where m is an odd integer $3 \le m \le n+2$, and $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ are real parameters to be determined later. Frequently, we consider the quadrature rule as

$$\mathcal{I}^m_{\mathbb{P}_2}(f) := \int_l \mathbb{P}_2 f(x)dx. \tag{1.10}$$

Considering method with coefficients (1.3), (1.4), (1.5), (1.6), (1.7), we have

$$E(f) = \mathcal{I}_{\mathbb{P}_2}(f) - \int_a^b f(x)dx = O(h^4) \quad \textit{low order}$$

and with coefficients (1.5), (1.6), (1.7), (1.8), (1.9),

$$E_m(f) = \mathcal{I}^m_{\mathbb{P}_2}(f) - \int_a^b f(x)dx = O(h^{m+1}). \quad \textit{high order}$$

A numerical comparison with the other rules shows the efficiency of this quadrature rule based on a quadratic spline quasi-interpolant.

2. Quadrature rules based on a quadratic spline quasi-interpolant

2.1. **Low order quadrature rule.** In this subsection, we review and describe the formulas for low order based on spline quasi-interpolant. Let $X_n := \{x_i,\ 0 \le i \le n\}$ be the uniform partition of the interval $I = [a, b]$ into n equal subintervals, i.e. $x_i := a + ih$, with $h = \frac{b-a}{n}$. We consider the space $S_2 = S_2(I, X_n)$ of quadratic splines of class C^1 on this partition. Canonical basis is formed by the $n+2$ normalized B-splines, $\{B_i,\ i \in J\}$, $J := \{1, 2, \cdots, n+2\}$. Consider the quadratic spline quasi-interpolant (dQI) of a function f defined on I and given in [16], that is

$$\mathbb{P}_2 f = \sum_{i \in J} \zeta_i(f) B_i, \tag{2.1}$$

where

$$\zeta_1(f) = f_1, \qquad \zeta_{n+2}(f) = f_{n+2},$$

$$\zeta_2(f) = -1/3 f_1 + 3/2 f_2 - 1/6 f_3 = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, \tag{2.2}$$

$$\zeta_{n+1}(f) = -1/6 f_n + 3/2 f_{n+1} - 1/3 f_{n+2} = \beta_3 f_n + \beta_2 f_{n+1} + \beta_1 f_{n+2}, \tag{2.3}$$

and for $3 \le j \le n$,

$$\zeta_j(f) = -1/8 f_{j-1} + 5/4 f_j - 1/8 f_{j+1} = \gamma_1 f_{j-1} + \gamma_2 f_j + \gamma_3 f_{j+1}, \tag{2.4}$$

with $f_i = f(t_i)$, $t_1 = a$, $t_{n+2} = b$, $t_i = a + (i - 3/2)h,\ 2 \le i \le n+1$. We consider the quadrature rule defined by

$$\mathcal{I}_{\mathbb{P}_2}(f) := \int_I \mathbb{P}_2 f(x)dx. \tag{2.5}$$

The quadratic B-spline basis functions at knots are defined as follows:

$$B_i(x)=\begin{cases}\dfrac{(x-x_{i-3})^2}{(x_{i-1}-x_{i-3})(x_{i-2}-x_{i-3})}, & x_{i-3}\le x<x_{i-2},\\[2ex] \dfrac{(x_i-x)(x-x_{i-2})}{(x_i-x_{i-2})(x_{i-1}-x_{i-2})}+\dfrac{(x-x_{i-3})(x_{i-1}-x)}{(x_{i-1}-x_{i-3})(x_{i-1}-x_{i-2})}, & x_{i-2}\le x<x_{i-1},\\[2ex] \dfrac{(x_i-x)^2}{(x_i-x_{i-2})(x_i-x_{i-1})}, & x_{i-1}\le x<x_i.\end{cases}$$

By using $\int_a^b B_j dx=\frac{1}{3}(x_j-x_{j-3})$, we can get

$$\int_I B_1(x)dx=\int_I B_{n+2}(x)dx=h/3,$$

$$\int_I B_2(x)dx=\int_I B_{n+1}(x)dx=2h/3,$$

$$\int_I B_k(x)dx=h,\qquad 3\le k\le n.$$

Quadrature formula $\mathcal{I}_{\mathbb{P}_2}$ can be written as

$$\begin{aligned}\mathcal{I}_{\mathbb{P}_2}(f)=\frac{h}{3}f_1+\frac{2h}{3}\sum_{i=1}^{3}\beta_i(f_i+f_{n+3-i})+h\sum_{i=3}^{n}(\gamma_1 f_{i-1}+\gamma_2 f_i+\gamma_3 f_{i+1})\\ +\frac{h}{3}f_{n+2}.\end{aligned}\tag{2.6}$$

2.2. **High order quadrature rule.** For $I=[a,b]$, we consider $S_2(I,X_n)$ the space of splines of degree 2 and class C^1 on uniform partition $X_n=\{x_i=a+ih, 0\le i\le n\}, h=\frac{b-a}{n}$. A basis of this space is $\{B_j, j=1,2,\dots,n+2\}$. Considering $\mathbb{P}_2 f=\sum_{i\in J}\zeta_i(f)B_i$.

Now, we will make quadrature rules based on this type of dQI and of high order convergence. For $j=2,\dots,n+1$, we appoint the same values of $\zeta_j(f)$, defined by (2.2), (2.3), (2.4) and for $j=1$ and $j=n+2$,

$$\zeta_1(f)=\sum_{i=1}^{m}\alpha_i f_i,\tag{2.7}$$

$$\zeta_{n+2}(f)=\sum_{i=1}^{m}\alpha_i f_{n+3-i},\tag{2.8}$$

where m is an odd integer $3\le m\le n+2$, and $\{\alpha_i\}_{i=1}^m$ are real values to be determined later. Frequently, we consider the quadrature rule as

$$\mathcal{I}_{\mathbb{P}_2}^m(f):=\int_I \mathbb{P}_2 f(x)dx.\tag{2.9}$$

We suppose error of the quadrature rule (2.9) on function x^{m-1} with $I = [0, n]$ and $h = 1$ as

$$
\begin{aligned}
E_m &= \mathcal{I}^m_{\mathbb{P}_2}(f) - \int_a^b f(x)dx \\
&1/3 \left(\sum_{i=1}^{m} \alpha_i (t_i^{m-1} + t_{n+3-i}^{m-1}) \right) + 2/3 \left(\sum_{i=1}^{3} \beta_i (t_i^{m-1} + t_{n+3-i}^{m-1}) \right) \\
&+ \sum_{j=3}^{n} (\gamma_1 t_{j-1}^{m-1} + \gamma_2 t_j^{m-1} + \gamma_3 t_{j+1}^{m-1}) - \frac{1}{m} n^m.
\end{aligned}
$$

This quadrature formula with m correction points and based on integrating quasi-interpolant $\mathbb{P}_2$ can be given as

$$
\mathcal{I}^m_{\mathbb{P}_2}(f) = h \sum_{i=1}^{m} \eta_i^{(m,2)} (f_i + f_{n+3-i}) + h \sum_{i=m+1}^{n+2-m} f_i. \tag{2.10}
$$

In Table 12, we give correction weights $\{\eta_i^{(m,2)}, i = 1, \dots, m\}$. Now we obtain the values of $\{\alpha_i\}_{i=1}^m$. For this purpose, we need the following Lemmas.

Lemma 2.1. *Let m be an odd integer with $3 \le m \le n+2$, and let*

$$
S = \sum_{j=3}^{n} (\gamma_1 t_{j-1}^{m-1} + \gamma_2 t_j^{m-1} + \gamma_3 t_{j+1}^{m-1}).
$$

Then S is a polynomial function of degree m in the variable n. More precisely,

$$
S = \sum_{j=0}^{m} \theta_j^{(m)} n^j, \tag{2.11}
$$

where

$$
\begin{aligned}
\theta_0^{(m)} &= \sum_{l=1}^{3} \gamma_l \Big(\frac{1}{m} ((l - 7/2)^m + (1/2 - l)^m) + 1/2((l - 7/2)^{m-1} + (1/2 - l)^{m-1}) \\
&+ \sum_{i=1}^{\frac{m-1}{2}} \frac{\tilde{B}_{2i}}{2i!} (m-1) \dots (m-2i+1)((l - 7/2)^{m-2i} + (1/2 - l)^{m-2i}) \Big),
\end{aligned}
$$

$$
\begin{aligned}
\theta_j^{(m)} &= \sum_{l=1}^{3} \gamma_l \Big(\frac{1}{m} C_m^j (l - 7/2)^{m-j} + 1/2 C_{m-1}^j (l - 7/2)^{m-1-j} \\
&+ \sum_{i=1}^{[\frac{m-j-1}{2}]} \frac{\tilde{B}_{2i}}{2i!} (m-1) \dots (m-2i+1) C_{m-2i}^j (l - 7/2)^{m-2i-j} \Big), j = 1, \dots, m,
\end{aligned}
$$

where C_m^j are the binomial coefficients, $\tilde{B}_{2i}$ are the Bernoulli numbers and $[x]$ denotes the integer part of x.

Proof. For proof, refer to [17]. □

Lemma 2.2. *For m odd and $3 \leq m \leq n+2$, we have*

$$E_m = \sum_{j=0}^{m-1} \lambda_j^{(m)} n^j,$$

where

$$\lambda_0^{(m)} = 2(\frac{1}{3}\sum_{i=1}^{m} \alpha_i t_i^{m-1} + \frac{2}{3}\sum_{i=1}^{3} \beta_i t_i^{m-1}) + \theta_0^m,$$

$$\lambda_j^{(m)} = \frac{1}{3}\sum_{i=1}^{m} \alpha_i C_{m-1}^j (-t_i)^{m-1-j} + \frac{2}{3}\sum_{i=1}^{3} \beta_i C_{m-1}^j (-t_i)^{m-1-j} + \theta_j^{(m)}.$$

According to the Lemma 1, we deduce that imposing $E_m = 0$ for all n is equivalent to solving the following linear system on $(\alpha_1, \alpha_2, \ldots, \alpha_m)$:

$$\lambda_j^{(m)} = 0, \qquad j = 0, \ldots, m-1. \tag{2.12}$$

Proof. For proof, refer to [17]. □

Having used the Lemma 2.1 and 2.2, we can get the parameters $\alpha_1, \ldots, \alpha_m$ as Table 1.

TABLE 1. The values of parameters α_i

α_i	$m=5$	$m=7$	$m=9$	$m=13$
α_1	1.05397	1.08045	1.09145	1.09745
α_2	−0.108854	−0.177438	−0.210955	−0.232091
α_3	0.0904514	0.193017	0.264956	0.327069
α_4	−0.0432292	−0.150553	−0.267354	−0.41376
α_5	0.00766369	0.0723562	0.19614	0.4412
α_6	−	−0.0204115	−0.105158	−0.402043
α_7	−	0.00257848	0.0386972	0.302815
α_8	−	−	−0.00866666	−0.182146
α_9	−	−	0.000886433	0.0846578
α_{10}	−	−	−	−0.0291921
α_{11}	−	−	−	0.00701754
α_{12}	−	−	−	−0.00104865
α_{13}	−	−	−	0.0000733031

3. Application to Hammerstein integral equations

In this section, we illustrate an application of the quadrature rules low and high orders to numerical solution of hammerstein integral equation (1.1).

3.1. **Approximate solution using low order quadrature rule.** First, we apply the low order quadrature rule for obtaining the approximate solution of Hammerstein integral equation (1.1). Considering Eq. (1.1) and using Eq. (2.6) we can get

$$\begin{aligned}
&u^{(n)}(x) - (\frac{h}{3}k(x,t_1)\mathfrak{F}(t_1,u^{(n)}(t_1)) + \frac{2h}{3}\sum_{j=1}^{3}\beta_j(k(x,t_j)\mathfrak{F}(t_j,u^{(n)}(t_j))\\
&+ k(x,t_{n+3-j})\mathfrak{F}(t_{n+3-j},u^{(n)}(t_{n+3-j})))\\
&+ h\sum_{j=3}^{n}(\gamma_1 k(x,t_{j-1})\mathfrak{F}(t_{j-1},u^{(n)}(t_{j-1})) + \gamma_2 k(x,t_j)\mathfrak{F}(t_j,u^{(n)}(t_j)) +\\
&\gamma_3 k(x,t_{j+1})\mathfrak{F}(t_{j+1},u^{(n)}(t_{j+1}))) + \frac{h}{3}k(x,t_{n+2})\mathfrak{F}(t_{n+2},u^{(n)}(t_{n+2}))) = g(x),
\end{aligned} \tag{3.1}$$

where $u^{(n)}(x)$ is approximate solution. We can rewrite the Equation (3.1) as follows:

$$u^{(n)}(x) - \sum_{j=1}^{n+2} w_j k(x,t_j)\mathfrak{F}(t_j,u_j^{(n)}) = g(x), \tag{3.2}$$

where

$$w_j = \begin{cases} \frac{h}{3} + \frac{2h}{3}\beta_1, & j = 1, n+2,\\ \frac{2h}{3}\beta_2 + h\gamma_1, & j = 2,\\ \frac{2h}{3}\beta_3 + h\gamma_2 + h\gamma_1, & j = 3,\\ h\gamma_1 + h\gamma_2 + h\gamma_3, & 4 \le j \le n-1,\\ \frac{2h}{3}\beta_3 + h\gamma_2 + h\gamma_3, & j = n,\\ \frac{2h}{3}\beta_2 + h\gamma_3, & j = n+1. \end{cases}$$

Substituting $x = x_i, i = 1,\ldots,n+2$ in the system (3.2) can be simplified in the matrix form

$$U_n - K_n^*\mathrm{F} = G, \tag{3.3}$$

where

$$\begin{aligned}
&U_n = [u_1^{(n)}, u_2^{(n)}, \ldots, u_{n+2}^{(n)}]^T,\\
&K_n^* = [w_j k(x_i,t_j)]_{i,j}, i,j = 1,\ldots,n+2,\\
&\mathrm{F} = [\mathfrak{F}(t_1,u_1^{(n)}), \mathfrak{F}(t_2,u_2^{(n)}), \ldots, \mathfrak{F}(t_{n+2},u_{n+2}^{(n)})]^T,\\
&G = [g(x_1), g(x_2), \ldots, g(x_{n+2})]^T.
\end{aligned}$$

The above nonlinear system consists of $n+2$ equations with $n+2$ unknown coefficients $\{u_j^{(n)}\}_{j=1}^{n+2}$. Solving this nonlinear system by Newton's method, we can obtain the values of $\{u_j^{(n)}\}_{j=1}^{n+2}$. Having used the solution $\{u_j^{(n)}\}_{j=1}^{n+2}$, we employ a method similar to the Nyström's idea for the Hammerstein integral equation, i.e. we use

$$u^{(n)}(x) = \sum_{j=1}^{n+2} w_j k(x, t_j)\mathrm{F}(t_j) + g(x). \tag{3.4}$$

3.2. **Approximate solution using high order quadrature rule.** In this subsection, we use spline quasi-interpolant method of high convergence order for approximate solution of Hammerstein integral equation (1.1). By using Eq. (2.10) and full simplifying, the integral part of Eq. (1.1) can be reduced as

$$\int_a^b k(x,t)\mathfrak{F}(t,u(t))ds = \sum_{j=1}^{n+2} w_j^{(m)} k(x,t_j)\mathfrak{F}(t_j, u_j^{(m)}), \tag{3.5}$$

where

$$w_j^{(m)} = \begin{cases} h\eta_j^{(m,2)}, & j = 1, \dots, m, \\ h, & j = m+1, \dots, n-m+2, \\ h\eta_{n+3-j}^{(m,2)}, & j = n-m+3, \dots, n+2. \end{cases}$$

Having applied (3.5) we have

$$u^{(m)}(x) - \sum_{j=1}^{n+2} w_j^{(m)} k(x,t_j)\mathfrak{F}(t_j, u_j^{(m)}) = g(x). \tag{3.6}$$

By replacing $x = x_i, i = 1, \dots, n+2$ we can get

$$u^{(m)}(x_i) - \sum_{j=1}^{n+2} w_j^{(m)} k(x_i, t_j)\mathrm{F}(t_j) = g(x_i), \tag{3.7}$$

where $\mathrm{F}(t_j) = \mathfrak{F}(t_j, u_j^{(m)})$. The system of nonlinear equations (3.7) for unknown $\{u_j^{(m)}\}_{j=1}^{n+2}$, can be expressed in a matrix form

$$U_{n,m} - K_{n,m}^{*}\mathrm{F} = G, \tag{3.8}$$

where

$$\begin{aligned} &U_{n,m} = [u_1^{(m)}, u_2^{(m)}, \dots, u_{n+2}^{(m)}]^T, \\ &K_{n,m}^{*} = [w_j^{(m)} k(x_i, t_j)]_{i,j}, i, j = 1, \dots, n+2, \\ &\mathrm{F} = [\mathfrak{F}(t_1, u_1^{(m)}), \mathfrak{F}(t_2, u_2^{(m)}), \dots, \mathfrak{F}(t_{n+2}, u_{n+2}^{(m)})]^T, \\ &G = [g(x_1), g(x_2), \dots, g(x_{n+2})]^T. \end{aligned}$$

The above nonlinear system consists $n+2$ equations with $n+2$ unknown coefficients $\{u_j^{(m)}\}_{j=1}^{n+2}$. Solving this nonlinear system by Newton's method, we can obtain the values of $\{u_j^{(m)}\}_{j=1}^{n+2}$. Having used the solution $\{u_j^{(m)}\}_{j=1}^{n+2}$, we employ a method similar to the Nyström's idea for the Hammerstein integral equation, i.e. we use

$$u^{(m)}(x) = \sum_{j=1}^{n+2} w_j^{(m)} k(x,t_j)\mathrm{F}(t_j) + g(x). \tag{3.9}$$

Each step of Newton iteration step involves evaluation of the vectors $A_n^{(l)}$ and $A_{n,m}^{(l)}$, the Jacobian matrix $J_n^{(l)}$, $J_{n,m}^{(l)}$ and $\Delta U_n^{(l)}$ and $\Delta U_{n,m}^{(l)}$. whenever the distance between two iterations is less than a given tolerance, ϵ, then the algorithm is to stop.

$$\| U_n^{(l+1)} - U_n^{(l)} \| \leq \epsilon, \ \| U_{n,m}^{(l+1)} - U_{n,m}^{(l)} \| \leq \epsilon.$$

Algorithm 1:

intialize: $U_n = U_n^{(0)}$ $(U_{n,m} = U_{n,m}^{(0)})$
for $l = 0, 1, 2, \ldots$
$A_n^{(l)} = U_n|^{(l)} - K_n^* F|^{(l)} - G$ $(A_{n,m}^{(l)} = U_{n,m}|^{(l)} - K_{n,m}^* F|^{(l)} - G)$
if $\| A_n^{(l)} \|$ $(\| A_{n,m}^{(l)} \|)$ is small enough, stop
compute $J_n^{(l)}$ $(J_{n,m}^{(l)})$
solve $J_n^{(l)} \Delta U_n^{(l)} = -A_n(U_n^{(l)})$ $(J_{n,m}^{(l)} \Delta U_{n,m}^{(l)} = -A_{n,m}(U_{n,m}^{(l)}))$
$U_n^{(l+1)} = U_n^{(l)} + \Delta U_n^{(l)}$ $(U_{n,m}^{(l+1)} = U_{n,m}^{(l)} + \Delta U_{n,m}^{(l)})$
end

4. Convergence Analysis

In this section, we shall provide the convergence analysis of the proposed methods. For this purpose, consider the following theorems. According [18], for any function $f \in C^3(I)$ we have

$$\| f - \mathbb{P}_2 f \|_\infty \leq C h^3 \| f^{(3)} \|_\infty .$$

We consider $\mathfrak{E}(f,z)$ the error of the quadrature formula based on $\mathbb{P}_2$ with smooth weight function z as

$$\mathfrak{E}(f,z) = \int_0^1 (f - \mathbb{P}_2 f) z. \tag{4.1}$$

Theorem 4.1. *Assume that $d+1$ is odd and let $z \in W_1^1(I)$, where $W_1^1(I)$ is the Sobolev space of functions with integrable derivatives. Then we have*

$$\mathfrak{E}_z(f) = O(h^4). \tag{4.2}$$

Proof. For proof, refer to [3]. □

Theorem 4.2. *Let $\tilde{r}_n$ error term for the spline quasi-interpolant method. Furthermore, let $M_0 = \max |w_j||k(x_i,t_j)|$ and $\chi_j = \max |k(x_i,t_j)|$. Then*

$$|\epsilon_{n,i}| \le \frac{O(h^4)}{1-M_0} exp(\sum_{j=1}^{n+1} \frac{\chi_j}{1-M_0}). \tag{4.3}$$

Proof. In fact

$$u^{(n)}(x_i) = \sum_{j=1}^{n+2} w_j k(x_i,t_j)\mathfrak{F}(t_j, u^{(n)}(t_j)) + g(x_i).$$

Thus

$$\begin{aligned} u^{(n)}(x_i) - u(x_i) = &\sum_{j=1}^{n+2} w_j k(x_i,t_j)(\mathfrak{F}(t_j,u^{(n)}(t_j)) - \mathfrak{F}(t_j,u(t_j))) \\ &+ \sum_{j=1}^{n+2} w_j k(x_i,t_j)\mathfrak{F}(x_i,u(x_i)) - \int_a^b k(x_i,s)\mathfrak{F}(s,u(s))ds. \end{aligned} \tag{4.4}$$

Let

$$\epsilon_{n,i} = u^{(n)}(x_i) - u(x_i), \tag{4.5}$$

and

$$\tilde{r}_n(x_i) = \sum_{j=1}^{n+2} w_j k(x_i,t_j)\mathfrak{F}(x_i,u(x_i)) - \int_a^b k(x_i,s)\mathfrak{F}(s,u(s))ds.$$

Hence

$$\epsilon_{n,i} = \sum_{j=1}^{n+2} w_j k(x_i,t_j)\epsilon_{n,j} + \tilde{r}_n(x_i).$$

Thus

$$|\epsilon_{n,i}| \le \sum_{j=1}^{n+2} |w_j||k(x_i,t_j)||\epsilon_{n,j}| + |\tilde{r}_n(x_i)|,$$

so that

$$|\epsilon_{n,i}| \le \sum_{j=1}^{n+1} |w_j||k(x_i,t_j)||\epsilon_{n,j}| + |w_j k(x_i,t_n)||\epsilon_{n,n}| + |\tilde{r}_n(x_i)|.$$

Now using the Gronwall Lemma [6], we obtain

$$|\epsilon_{n,i}| \le \frac{O(h^4)}{1-M_0} exp(\sum_{j=1}^{n+1} \frac{\chi_j}{1-M_0}).$$

□

Now we discuss the convergence analysis for high order quadrature rule. For this purpose, we need following theorem.

Theorem 4.3. *Suppose that m is an odd integer $3 \leq m \leq n+2$, and (a,b) is a pair of real numbers such that $a < b$. Let $(\alpha_1, \alpha_2, \dots, \alpha_m)$ be a solution of the linear system (2.12). Then the quadrature rule $\mathcal{I}^m_{\mathbb{P}_2}$ given by (2.10), is of order $m+1$, i.e. for any $f \in C^{m+1}[a,b]$, there exists a real number $c > 0$ independent of n, such that*

$$\left| \mathcal{I}^m_{\mathbb{P}_2}(f) - \int_a^b f(x)dx \right| < \frac{c}{n^{m+1}}. \tag{4.6}$$

Furthermore, the dQI $\mathbb{P}_2$ given by (1.2) with modified functionals (2.7), (2.8) is exact on π_2.

Proof. For proof, refer to [17]. □

Now, we can prove the following theorem, which shows that the high order quadrature method convergence at the rate of $O(h^{m+1})$.

Theorem 4.4. *Let $\tilde{r}_n$ error term for the spline quasi-interpolant method. Furthermore, let $M_1 = \max |w_j^{(m)}||k(x_i,t_j)|$ and $\psi_j = \max |k(x_i,t_j)|$. Then*

$$|\epsilon_{n,i}| \leq \frac{O(h^{m+1})}{1-M_1} exp(\sum_{j=1}^{n+1} \frac{\psi_j}{1-M_1}). \tag{4.7}$$

Proof. In fact

$$u^{(m)}(x_i) = \sum_{j=1}^{n+2} w_j^{(m)} k(x_i,t_j) \mathfrak{F}(t_j, u^{(m)}(t_j)) + g(x_i).$$

Thus

$$\begin{aligned} u^{(m)}(x_i) - u(x_i) &= \sum_{j=1}^{n+2} w_j^{(m)} k(x_i,t_j)(\mathfrak{F}(t_j,u^{(m)}(t_j)) - \mathfrak{F}(t_j,u(t_j))) \\ &+ \sum_{j=1}^{n+2} w_j k(x_i,t_j)\mathfrak{F}(x_i,u(x_i)) - \int_a^b k(x_i,s)\mathfrak{F}(s,u(s))ds. \end{aligned} \tag{4.8}$$

Let

$$\epsilon_{n,i} = u^{(m)}(x_i) - u(x_i), \tag{4.9}$$

and

$$\tilde{r}_n(x_i) = \sum_{j=1}^{n+2} w_j^{(m)} k(x_i,t_j)\mathfrak{F}(x_i,u(x_i)) - \int_a^b k(x_i,s)\mathfrak{F}(s,u(s))ds.$$

Hence

$$\epsilon_{n,i} = \sum_{j=1}^{n+2} w_j^{(m)} k(x_i,t_j)\epsilon_{n,j} + \tilde{r}_n(x_i).$$

Thus

$$|\epsilon_{n,i}| \leq \sum_{j=1}^{n+2} |w_j^{(m)}||k(x_i,t_j)||\epsilon_{n,j}| + |\tilde{r}_n(x_i)|,$$

so that

$$|\epsilon_{n,i}| \leq \sum_{j=1}^{n+1} |w_j^{(m)}||k(x_i,t_j)||\epsilon_{n,j}| + |w_j^{(m)}k(x_i,t_n)||\epsilon_{n,n}| + |\tilde{r}_n(x_i)|.$$

Now using the Gronwall Lemma [6], we obtain

$$|\epsilon_{n,i}| \leq \frac{O(h^{m+1})}{1-M_1} exp(\sum_{j=1}^{n+1} \frac{\psi_j}{1-M_1}).$$

□

5. Numerical Results

In this section, in order to illustrate the performance of the presented methods in solving Hammerstein integral equations and justify the accuracy and efficiency of the methods, we consider the following examples.

Example 5.1. Consider the following Hammerstein integral equation

$$u(x) = -\int_0^1 e^{x-2t}u^3(t)dt + e^{x+1},$$

where the exact solution is $u(x) = e^x$. In Table 2, numerical results are presented for rule $\mathcal{I}_{\mathbb{P}_2}(f)$. We obtain an approximation to the solution of Eq. (3.4). Also in Table 3, numerical results are presented for rules $\mathcal{I}_{\mathbb{P}_2}^m(f)$, $m = 5, 7, 9$, which we use Eq. (3.9) for obtaining solution. Numerical results illustrate accuracy of the proposed quadrature rules. By increasing the values of n the errors have been decreased. A numerical comparison between Tables $2, 3$ shows that spline quasi-interpolant method of high convergence order is better than of low convergence order. In Table 4, we compare the absolute errors of the spline quasi-interpolant method ($\mathcal{I}_{\mathbb{P}_2}^m$) with sinc method [14].

Table 2. Max. Abs. Err. for Example 5.1 ($\mathcal{I}_{\mathbb{P}_2}$)

n	$\parallel u - u_n \parallel_\infty$
15	5.42814×10^{-8}
20	1.76127×10^{-8}
25	7.32217×10^{-9}
30	3.56595×10^{-9}
35	1.93825×10^{-9}
40	1.14209×10^{-9}
45	5.15875×10^{-10}
50	4.71196×10^{-10}
55	3.22678×10^{-10}

Table 3. Max. Abs. Err. for Example 5.1 ($\mathcal{I}_{\mathbb{P}_2}^m$)

n	$m=5$	$m=7$	$m=9$
20	9.99378×10^{-12}	7.99361×10^{-15}	8.88178×10^{-16}
25	2.71916×10^{-12}	2.22045×10^{-15}	8.88178×10^{-16}
30	9.33031×10^{-13}	1.33227×10^{-15}	8.88178×10^{-16}
35	3.76588×10^{-13}	8.8817×10^{-16}	1.33227×10^{-15}
40	1.71418×10^{-13}	1.77636×10^{-15}	1.33227×10^{-15}
45	8.57092×10^{-14}	8.88178×10^{-16}	8.888178×10^{-16}
50	4.61853×10^{-14}	8.88178×10^{-16}	1.77636×10^{-15}
55	2.66454×10^{-14}	8.88178×10^{-16}	8.88178×10^{-16}

Table 4. Numerical results for Example 5.1

n	$m=5$	[14]
15	5.26406×10^{-11}	2.05487×10^{-6}
25	2.71916×10^{-12}	3.46962×10^{-8}
35	3.76588×10^{-13}	9.73006×10^{-10}
45	8.57092×10^{-14}	4.20691×10^{-11}
55	2.66454×10^{-14}	4.85937×10^{-12}

Example 5.2. We consider the integral equation

$$u(x) = \int_0^1 x.t\sqrt{u(t)}dt + 2 - \frac{1}{3}(2\sqrt{2}-1)x - x^2,$$

where the exact solution is $u(x) = 2 - x^2$. Numerical results are tabulated in Table 5 for Quadrature rule $\mathcal{I}_{\mathbb{P}_2}^m(f)$, $m = 5, 7, 9$. The table indicates that as n increases the errors decreases. In Table 6, we compare the absolute errors of the spline quasi-interpolant method for ($\mathcal{I}_{\mathbb{P}_2}^m$) $n = 20$ with the single exponential (SE)-Sinc method and double exponential (DE)-Sinc method [10]. Also in Table 7, we compare the absolute errors of the spline quasi-interpolant method with a novel numerical method [9]. Figure 1 shows the maximum absolute errors for the proposed method.

Table 5. Max. Abs. Err. for Example 5.2 ($\mathcal{I}_{\mathbb{P}_2}^m$)

n	$m=5$	$m=7$	$m=9$
25	1.26441×10^{-9}	4.53524×10^{-11}	2.59703×10^{-12}
35	1.96747×10^{-10}	4.33609×10^{-12}	1.63203×10^{-13}
45	4.77083×10^{-11}	7.10321×10^{-13}	1.88738×10^{-14}
55	1.51845×10^{-11}	1.62759×10^{-13}	3.10862×10^{-15}
65	5.80958×10^{-12}	4.70735×10^{-14}	6.66134×10^{-16}
75	2.53886×10^{-12}	1.59872×10^{-14}	2.22045×10^{-16}

TABLE 6. Numerical results for Example 5.2

x	$m=9$	(SE)-Sinc [10]	(DE)-Sinc [10]
0.1	1.48392×10^{-12}	3.08×10^{-6}	1.17×10^{-9}
0.3	4.45222×10^{-12}	2.92×10^{-6}	5.49×10^{-10}
0.5	7.42051×10^{-12}	2.03×10^{-6}	5.55×10^{-15}
0.7	1.03888×10^{-11}	2.58×10^{-6}	5.49×10^{-10}
0.9	1.33571×10^{-11}	2.41×10^{-6}	1.17×10^{-9}

TABLE 7. Numerical results for Example 5.2

x	$n=16$	$n=32$	$n=16$[9]	$n=32$[9]
0.2	1.55809×10^{-11}	6.92779×10^{-14}	4.09×10^{-4}	9.94×10^{-5}
0.4	3.11617×10^{-11}	1.38556×10^{13}	8.18×10^{-4}	1.99×10^{-4}
0.6	4.67426×10^{-11}	2.07834×10^{-13}	1.23×10^{-3}	2.98×10^{-4}
0.8	6.2323×10^{-11}	2.77112×10^{-13}	1.64×10^{-3}	3.97×10^{-4}

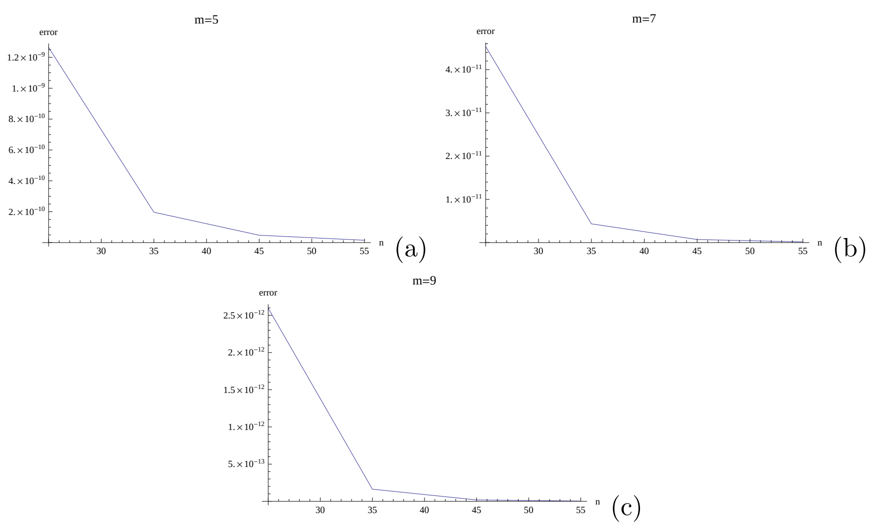

FIGURE 1. (a),(b),(c): The absolute error $\parallel u - u^{(m)} \parallel_\infty$ for $m=5, m=7, m=9$ and different values of n for Example 5.2.

Example 5.3. Consider the following Hammerstein integral equation

$$u(x) = \frac{1}{5}\int_0^1 \sin(\pi t)\cos(\pi x)u^3(t)dt + \sin(\pi x),$$

where the exact solution is $u(x) = \sin(\pi x) + \frac{1}{3}(20 - \sqrt{391})\cos(\pi x)$. In Table 8, numerical results are presented for rules $\mathcal{I}^m_{\mathbb{P}_2}(f)$, $m = 5, 7, 9$. Numerical results illustrate accuracy of the proposed quadrature rule. By increasing the values of n the errors have been decreased. In Table 9, we compare the absolute errors of the spline quasi-interpolant method for $n = 25$ with Newton-Kantorovich-quadrature (NKQ) method [15], the single exponential (SE)-Sinc method and double exponential (DE)-Sinc method [10]. Figure 2 shows the maximum absolute errors for the proposed method.

Table 8. Max. Abs. Err. for Example 5.3 ($\mathcal{I}^m_{\mathbb{P}_2}$)

n	$m = 5$	$m = 7$	$m = 9$
20	1.78317×10^{-7}	3.00843×10^{-8}	3.53921×10^{-9}
30	1.16704×10^{-8}	1.04487×10^{-9}	7.76529×10^{-11}
40	1.61868×10^{-9}	8.62851×10^{-11}	3.96863×10^{-12}
50	3.45495×10^{-10}	1.20916×10^{-11}	3.70579×10^{-13}
60	9.7345×10^{-11}	2.39866×10^{-12}	5.19168×10^{-14}
70	3.32794×10^{-11}	6.07625×10^{-13}	9.5618×10^{-15}

Table 9. Numerical results for Example 5.3

x	$m = 9$	NKQ [15]	(SE)-Sinc [10]	(DE)-Sinc [10]
0.1	4.46313×10^{-10}	4.73×10^{-2}	1.33×10^{-5}	1.29×10^{-8}
0.2	3.79657×10^{-10}	4.03×10^{-2}	6.85×10^{-6}	2.05×10^{-8}
0.3	2.75837×10^{-10}	2.92×10^{-2}	6.22×10^{-6}	1.25×10^{-8}
0.4	1.45016×10^{-10}	1.54×10^{-2}	2.19×10^{-6}	2.41×10^{-9}
0.5	0.0000	0.0000	2.15×10^{-7}	0.0000
0.6	1.45016×10^{-10}	1.54×10^{-2}	5.85×10^{-6}	9.54×10^{-9}
0.7	2.75837×10^{-10}	2.92×10^{-2}	1.02×10^{-5}	2.31×10^{-8}
0.8	3.79657×10^{-10}	4.03×10^{-2}	8.25×10^{-6}	2.33×10^{-8}
0.9	3.36599×10^{-10}	4.73×10^{-2}	1.66×10^{-5}	1.61×10^{-8}

Example 5.4. Consider the following Hammerstein integral equation

$$u(x) = x \int_0^1 \frac{4t + \pi \sin(\pi t)}{u^2(t) + t^2 + 1} - 2x \log(3) + \sin(\frac{\pi}{2}x).$$

In Table 10, approximate solutions are presented for rule $\mathcal{I}^m_{\mathbb{P}_2}(f)$, $m = 9, n = 20$.

Example 5.5. Consider the following Hammerstein integral equation

$$u(x) = \int_0^1 2xtexp(u^2(t))dt + \frac{x}{exp(1)}.$$

In Table 11, approximate solutions are presented for rule $\mathcal{I}^m_{\mathbb{P}_2}(f)$, $m = 9, n = 50$.

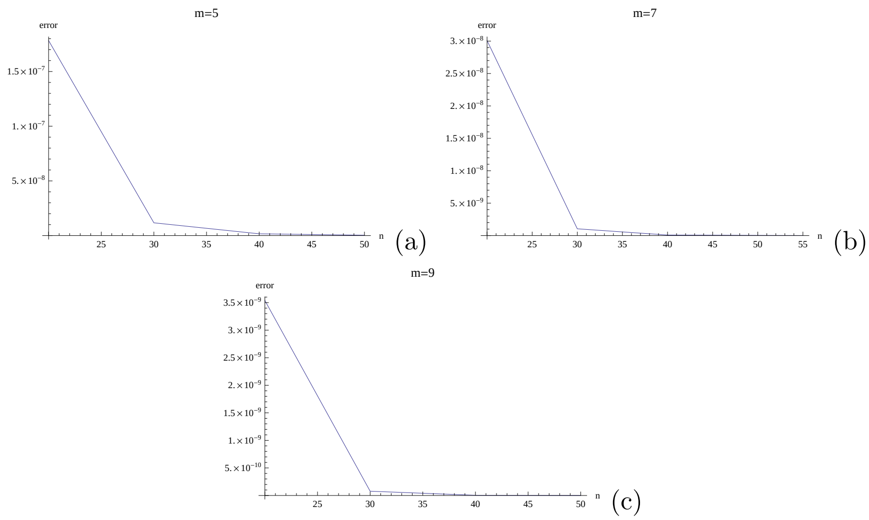

FIGURE 2. (a),(b),(c): The absolute error $\| u - u^{(m)} \|_\infty$ for $m = 5, m = 7, m = 9$ and different values of n for Example 5.3.

TABLE 10. Numerical solutions for Example 5.4

x	approximate solution
0.1	0.156434
0.2	0.309017
0.3	0.45399
0.4	0.587785
0.5	0.707107
0.6	0.809017
0.7	0.891007
0.8	0.951057
0.9	0.987688

6. CONCLUSION

The spline quasi-interpolant quadrature rules $\mathcal{I}_{\mathbb{P}_2}$ and $\mathcal{I}^m_{\mathbb{P}_2}$ are used to solve the Hammerstein integral equation. The convergence analysis of the presented methods are discussed. Also the results obtained here were compared with exact solution and methods in [9, 10, 14, 15]. The methods are computationally attractive and applications are demonstrate through illustrative examples.

TABLE 11. Numerical solutions for Example 5.5

x	approximate solution
0.1	0.10000
0.2	0.20000
0.3	0.30000
0.4	0.40000
0.5	0.50000
0.6	0.60000
0.7	0.70000
0.8	0.80000
0.9	0.90000

TABLE 12. Quadrature weights $\eta_i^{(m,2)}$

$m = 5$	$m = 7$	$m = 9$	$m = 13$
0.1307936	0.1374149	0.1400901	0.1414888
0.8359375	0.8190165	0.8109525	0.8060753
1.0449652	1.0698175	1.0870467	1.1014595
0.9861458	0.9603402	0.9321887	0.8977344
1.0021577	1.0177210	1.0478285	1.1060041
–	0.9950634	0.9742957	0.9037215
–	1.0006252	1.0095197	1.0720481
–	–	0.9978581	0.9571099
–	–	1.0002198	1.0196830
–	–	–	0.9933074
–	–	–	1.0015852
–	–	–	0.9997667
–	–	–	1.00001605

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